

$\mathcal{N}=4$ superparticle and super Yang-Mills theory in $USp(4)$ harmonic superspace

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Abstract

We study the $\mathcal{N}=4$ harmonic superparticle model, both with and without central charge and quantize it. Since the central charge breaks the $U(4)$ R-symmetry group of the $\mathcal{N}=4$ superalgebra down to $USp(4)$, we consider the superparticle dynamics in $\mathcal{N}=4$ harmonic superspace with $USp(4)/(U(1) \times U(1))$ harmonic variables. We show that the quantization of a massive superparticle with central charge leads to a superfield realization of the $\mathcal{N}=4$ massive vector multiplet in $\mathcal{N}=4$ harmonic superspace. In the massless case without central charge the superparticle quantization reproduces three different multiplets: the $\mathcal{N}=4$ SYM multiplet, the $\mathcal{N}=4$ gravitino multiplet and $\mathcal{N}=4$ supergravity multiplet. The SYM multiplet is described by six analytic superfield strengths with different types of analyticity. We show that these strengths solve the $\mathcal{N}=4$ SYM constraints and can be used for the construction of actions in $\mathcal{N}=4$ harmonic superspace.

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1 Introduction

The $\mathcal{N}=4$ super Yang-Mills (SYM) field theory, being the maximally extended rigid supersymmetric model, possesses many remarkable properties. The symmetry of this model is so large that the only freedom in the classical action is the choice of the gauge group, and the quantum dynamics is free of divergences. It worth pointing out that this theory has profound relations with superstring theory, particularly due to the AdS/CFT correspondence (see, e.g., [1]).

The problems of $\mathcal{N}=4$ SYM theory in the quantum domain are mainly related to the effective action and correlation functions of composite operators. The superfield approaches seem to be more efficient for these purposes, since they allow one to use the supersymmetries in explicit form. However, a description of $\mathcal{N}=4$ SYM theory in terms of unconstrained $\mathcal{N}=4$ superfields is still missing. For various applications, formulations in terms of $\mathcal{N}=1$ superfields (see, e.g., [2]), in terms of $\mathcal{N}=2$ superfields [3, 4], or in terms of $\mathcal{N}=3$ superfields [5] are used. All attempts to find an unconstrained $\mathcal{N}=4$ harmonic superfield formulation for the $\mathcal{N}=4$ SYM theory have been futile so far [7, 8, 9, 10, 11, 12], for a number of different types of harmonic variables originating from various cosets of the $SU(4)$ group. However, one may still hope that there exists some other harmonic superspace, not based on some $SU(4)$ coset, which is better suited for a superfield realization of $\mathcal{N}=4$ supergauge theory.² In other words, we need new superfield representations of the known irreducible multiplets of the $\mathcal{N}=4$ superalgebra realized in an appropriate harmonic superspace.

The main purpose of this paper is to construct $\mathcal{N}=4$ superparticle models in harmonic superspace, to quantize them and to derive superfield representations of the $\mathcal{N}=4$ superalgebra as a result of their quantization. It is well known that the quantization of superparticles is closely related to superfield formulations of the corresponding field theories. Indeed, the superparticle models are rich of symmetries such as reparameterization invariance, supersymmetry and, in particular cases, the kappa-symmetry (see, e.g., [13] for a review). All these symmetries are accompanied by constraints in the Hamiltonian formulation. Upon quantization, these constraints turn into equations of motion as well as superfield differential constraints, which together define superfield representations of irreducible multiplets of supersymmetry. For instance, the quantization of the $\mathcal{N}=1$ superparticle was achieved in [14, 15, 16, 17], the $\mathcal{N}=2$ gauge multiplet and hypermultiplets were obtained in [17, 18, 19, 20, 21] by quantizing the $\mathcal{N}=2$ superparticle and, finally, the $\mathcal{N}=3$ superparticle was recently studied by two of us [22], where massive and massless $\mathcal{N}=3$ vector multiplets as well as the $\mathcal{N}=3$ gravitino multiplet were derived. A particular case of massless superparticles with arbitrary $\mathcal{N}>2$ extended supersymmetry in $SU(\mathcal{N})$ harmonic superspace was analyzed in [18], where the corresponding superfield strengths were derived. We point out the significance of harmonic superparticles with $\mathcal{N}=2$ and $\mathcal{N}=3$ extended supersymmetries [18, 19, 22], since they yield equations of motion for the corresponding field theories in harmonic superspaces which possess unconstrained super-

²As is well known, an unconstrained superfield formulation of $\mathcal{N}=4$ SYM theory is impossible in standard $\mathcal{N}=4$ superspace.

field descriptions.

In the present paper we study models of the $\mathcal{N}=4$ harmonic superparticle both in the massive case with central charge and in the massless case without central charge. It is well known that a central charge breaks the $U(4)$ R-symmetry group of the $\mathcal{N}=4$ superalgebra down to $USp(4)$ [23, 24]. Therefore, we consider it as crucial to introduce $USp(4)$ harmonic variables which are employed for the corresponding $\mathcal{N}=4$ harmonic superspace. The various cosets of the $USp(4)$ group were introduced and studied in [25], and the corresponding harmonic variables were further applied in [26, 27] to $d=5$ and $d=6$ $\mathcal{N}=4$ SYM models. In our work we find them useful also for $d=4$ $\mathcal{N}=4$ SYM and superparticle models. We start with the formulation and quantization of the $\mathcal{N}=4$ superparticle in such a harmonic superspace and find superfield representations of various multiplets of the $\mathcal{N}=4$ superalgebra. In the massive case with nonzero central charge, the quantization leads to the massive $\mathcal{N}=4$ vector multiplet, represented by analytic superfields subject to several Grassmann and harmonic shortness conditions. In the massless case with vanishing central charge, this multiplet reduces to the usual $\mathcal{N}=4$ SYM multiplet if one also imposes reality conditions. As a result, the $\mathcal{N}=4$ SYM multiplet is described by six analytic superfields with different types of analyticity and harmonic shortness. Apart from the SYM multiplet, we also derive the $\mathcal{N}=4$ gravitino multiplet (with highest helicity $3/2$) and the multiplet of $\mathcal{N}=4$ supergravity. These multiplets are represented by $\mathcal{N}=4$ chiral-analytic and chiral superfields, respectively.

The quantization of the $\mathcal{N}=4$ superparticle appears to be very fruitful since it not only hints at a suitable $\mathcal{N}=4$ harmonic superspace based on $USp(4)$, but it also gives an appropriate formulation of $\mathcal{N}=4$ superfield strengths in such a harmonic superspace. We show that exactly these superfields appear in the solution of the $\mathcal{N}=4$ SYM constraints with the help of $USp(4)$ harmonic variables. Finally, we employ these superfields for constructing some integral invariants and sketch certain superfield actions which describe an F^4 term in such a harmonic superspace. Note that similar superfields were introduced in [8, 9, 10, 11, 18] by contracting the $\mathcal{N}=4$ superfield strengths with harmonics on some coset of $SU(4)$ and have been exploited for constructing $\mathcal{N}=4$ invariant actions and correlation functions of composite operators in various works (see, e.g., [12, 28]).

The paper is organized as follows. In Section 2 we introduce the $\mathcal{N}=4$ harmonic superspace with $USp(4)$ harmonic variables and review the basic constructions in it. In the next Section we consider the $\mathcal{N}=4$ harmonic superparticle model and develop the Lagrangian and Hamiltonian formulations for it. The quantization of the superparticle is given in Section 4, where the superfield representations for the massive and massless vector multiplets as well as for the gravitino and supergravity multiplets are found. In Section 5 we show how $USp(4)$ harmonic variables help solving the SYM constraints and construct various actions in such an $\mathcal{N}=4$ harmonic superspace. In a Summary we discuss the results obtained and some ideas of their further application to $\mathcal{N}=4$ SYM theory. Technical details are collected in three Appendices, where we address the problem of an unconstrained $\mathcal{N}=4$ superfield description of the F^2 term in harmonic superspace.

2 $\mathcal{N}=4$ $USp(4)$ harmonic superspace

2.1 $\mathcal{N}=4$ superalgebra with central charges

We start with a short review of $\mathcal{N}=4$ superspace and superalgebra constructions just to fix our notations. The generators of $\mathcal{N}=4$ superalgebra with central charges can be represented by the following differential operators

$$Q_\alpha^i = \frac{\partial}{\partial \theta_\alpha^i} - i \bar{\theta}^{\dot{\alpha}i} \sigma_{\alpha\dot{\alpha}}^m \frac{\partial}{\partial x^m} + i Z^{ij} \theta_{j\alpha}, \quad \bar{Q}_{\dot{\alpha}i} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}i}} + i \theta_i^\alpha \sigma_{\alpha\dot{\alpha}}^m \frac{\partial}{\partial x^m} + i \bar{Z}_{ij} \bar{\theta}_{\dot{\alpha}}^j, \quad (2.1)$$

where $\{x^m, \theta_{i\alpha}, \bar{\theta}_{\dot{\alpha}}^i\}$ are the superspace coordinates and $Z^{ij} = -Z^{ji}$, $\bar{Z}_{ij} = (Z^{ij})^*$ are constant antisymmetric matrices of central charges. The Greek letters $\alpha, \dot{\alpha}$ denote the $SL(2, C)$ indices while the small Latin ones $i, j, \dots = 1, 2, 3, 4$ correspond to R-symmetry. The operators (2.1) satisfy the standard anticommutation relations of $\mathcal{N}=4$ superalgebra with central charges,

$$\{Q_\alpha^i, Q_\beta^j\} = 2i \varepsilon_{\alpha\beta} Z^{ij}, \quad \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = -2i \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ij}, \quad \{Q_\alpha^i, \bar{Q}_{\dot{\alpha}j}\} = 2i \delta_j^i \sigma_{\alpha\dot{\alpha}}^m \frac{\partial}{\partial x^m}. \quad (2.2)$$

It is well known that without central charges the $\mathcal{N}=4$ superalgebra possesses $U(4)$ R-symmetry group. However, the non-zero central charges break the $U(4)$ R-symmetry group down to $USp(4)$ [23, 24]. Indeed, the relations (2.2) are invariant under those $U(4)$ transformations with the matrices u^i_k which leave the antisymmetric constant tensor Z^{ij} invariant, $u^i_k u^j_l Z^{kl} = Z^{ij}$. Hence, Z^{ij} plays the role of invariant tensor in $Sp(4)$ group. The resulting the R-symmetry group is given by the intersection of $U(4)$ and $Sp(4)$ groups, that is nothing but $USp(4)$.

By applying the rotations with some unitary matrices $u^i_j \in U(4)$ to the central charge tensor Z^{ij} one can bring it to the normal form [24, 29],

$$Z^{ij} \rightarrow Z'^{ij} = u^i_k u^j_l Z^{kl} = \begin{pmatrix} 0 & -z_1 & 0 & 0 \\ z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z_2 \\ 0 & 0 & z_2 & 0 \end{pmatrix}, \quad (2.3)$$

where z_1, z_2 are non-negative numbers. We restrict ourself in the further considerations to the case $z_1 = z_2 \equiv z$, so we are left with the only central charge. More generally one can consider complex central charge z by giving arbitrary phase factor to it due to the $U(1)$ rotations in the $U(4)$ group. This is sufficient for obtaining short representations of $\mathcal{N}=4$ superalgebra with central charge, when the masses of multiplets are related with the central charges by the BPS condition [23, 24],

$$m^2 = z \bar{z}. \quad (2.4)$$

The central charge matrix (2.3) can be written as

$$Z^{ij} = z \Omega^{ij}, \quad \bar{Z}_{ij} = -\bar{z} \Omega_{ij}, \quad (2.5)$$

where

$$\Omega_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \Omega^{ij} = (\Omega_{ij})^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (2.6)$$

The matrix Ω will be considered further as the invariant tensor in the $USp(4)$ group.

The $\mathcal{N}=4$ superspace possesses the following supercovariant Cartan forms

$$\omega^M = \begin{cases} \omega^m &= dx^m - id\theta_i^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{i\dot{\alpha}} + i\theta_i^\alpha \sigma_{\alpha\dot{\alpha}}^m d\bar{\theta}^{i\dot{\alpha}}, \\ \omega_i^\alpha &= d\theta_i^\alpha, \\ \bar{\omega}^{i\dot{\alpha}} &= d\bar{\theta}^{i\dot{\alpha}}, \end{cases} \quad (2.7)$$

which will be further used in the construction of the superparticle Lagrangian.

2.2 $USp(4)$ harmonic variables

The harmonic variables on different cosets of the $USp(4)$ group were introduced in [25]. In the present work we are interested in the harmonics on the $USp(4)/(U(1) \times U(1))$ coset which we review in this subsection in some details.

The $USp(4)$ harmonic variables are 4×4 unitary matrices $u = (u^i_j)$ preserving the antisymmetric tensor Ω (2.6),

$$u \in USp(4) \quad \Rightarrow \quad u u^\dagger = 1, \quad u \Omega u^T = \Omega. \quad (2.8)$$

Note that it is not necessary to impose the constraint $\det u = 1$ since it follows from (2.8).

Let us denote the elements of complex conjugate matrix as $u^* = (\bar{u}_i^j)$.³ Then the identities (2.8) can be written for the matrix elements as

$$u^i_j \bar{u}_k^j = \delta_k^i, \quad u^i_j \Omega^{jk} u^l_k = \Omega^{il}. \quad (2.9)$$

As follows from (2.9),

$$\bar{u}_i^j = \Omega_{ik} u^k_l \Omega^{lj}, \quad (2.10)$$

the conjugated matrix in the $USp(4)$ group is not independent, but is expressed through the original one with the help of invariant tensor Ω . In other words, the fundamental and conjugated representations are equivalent, similarly as for the $SU(2)$ group. Hence, the invariant tensors Ω_{ij} and Ω^{ij} are used to lower and rise the $USp(4)$ indices, e.g.,

$$u^{ij} = u^i_k \Omega^{kj} = \Omega^{ik} \bar{u}_k^j, \quad \bar{u}_{ij} = \Omega_{ik} u^k_j = \bar{u}_i^k \Omega_{kj}. \quad (2.11)$$

Here we assume $(\Omega_{ij})^* = -\Omega^{ij}$.

³We assume that the complex conjugation flips the position of indices while the transposition changes their order, so that $u = (u^i_j)$, $u^* = (\bar{u}_i^j)$, $u^T = (u^i_i)$, $u^\dagger = (\bar{u}^j_i)$.

Now we introduce the $usp(4)$ algebra as a space spanned on the following differential operators

$$\begin{aligned}
S_1 &= D_1^1 - D_2^2, & S_2 &= D_3^3 - D_4^4, \\
D^{(++,0)} &= D_2^1, & D^{(--,0)} &= D_1^2, \\
D^{(0,++)} &= D_4^3, & D^{(0,--)} &= D_3^4, \\
D^{(+,+)} &= D_4^1 + D_2^3, & D^{(-,-)} &= D_3^2 + D_1^4, \\
D^{(+,-)} &= D_3^1 - D_2^4, & D^{(-,+)} &= D_4^2 - D_1^3,
\end{aligned} \tag{2.12}$$

where

$$D_j^i = u^i_k \frac{\partial}{\partial u^j_k}. \tag{2.13}$$

The commutation relations of the operators (2.12), given by (A.3), show that $D^{(++,0)}$, $D^{(0,++)}$, $D^{(+,+)}$, $D^{(-,+)}$ are rising operators, $D^{(--,0)}$, $D^{(0,--)}$, $D^{(-,-)}$, $D^{(+,-)}$ are lowering ones and S_1 , S_2 are Cartan generators in the $USp(4)$ group. The operators S_1 , S_2 measure the $U(1)$ charges of the generators of $USp(4)$ group,

$$[S_1, D^{(s_1, s_2)}] = s_1 D^{(s_1, s_2)}, \quad [S_2, D^{(s_1, s_2)}] = s_2 D^{(s_1, s_2)}. \tag{2.14}$$

It is convenient to label the harmonic variables by their $U(1)$ charges as well,

$$u^1_i = u_i^{(+,0)}, \quad u^2_i = u_i^{(-,0)}, \quad u^3_i = u_i^{(0,+)}, \quad u^4_i = u_i^{(0,-)}. \tag{2.15}$$

The harmonic derivatives (2.12) can now be rewritten in the more useful form for practical calculations with harmonics (2.15),

$$\begin{aligned}
S_1 &= u_i^{(+,0)} \frac{\partial}{\partial u_i^{(+,0)}} - u_i^{(-,0)} \frac{\partial}{\partial u_i^{(-,0)}}, & S_2 &= u_i^{(0,+)} \frac{\partial}{\partial u_i^{(0,+)}} - u_i^{(0,-)} \frac{\partial}{\partial u_i^{(0,-)}}, \\
D^{(\pm\pm,0)} &= u_i^{(\pm,0)} \frac{\partial}{\partial u_i^{(\mp,0)}}, & D^{(0,\pm\pm)} &= u_i^{(0,\pm)} \frac{\partial}{\partial u_i^{(0,\mp)}}, \\
D^{(\pm,\pm)} &= u_i^{(\pm,0)} \frac{\partial}{\partial u_i^{(0,\mp)}} + u_i^{(0,\pm)} \frac{\partial}{\partial u_i^{(\mp,0)}}, & D^{(\pm,\mp)} &= u_i^{(\pm,0)} \frac{\partial}{\partial u_i^{(0,\pm)}} - u_i^{(0,\mp)} \frac{\partial}{\partial u_i^{(\mp,0)}}.
\end{aligned} \tag{2.16}$$

Using the notations (2.15), the basic relations for harmonics (2.9) can be written as orthogonality

$$u^{(+,0)i} u_i^{(-,0)} = u^{(0,+)i} u_i^{(0,-)} = 1, \tag{2.17}$$

$$u_i^{(+,0)} u^{(0,+)i} = u_i^{(+,0)} u^{(0,-)i} = u_i^{(0,+)i} u^{(-,0)i} = u_i^{(-,0)i} u^{(0,-)i} = 0 \tag{2.18}$$

and completeness conditions,

$$u^{(+,0)i} u_j^{(-,0)} - u_j^{(+,0)} u^{(-,0)i} + u^{(0,+)i} u_j^{(0,-)} - u_j^{(0,+)i} u^{(0,-)i} = \delta_j^i. \tag{2.19}$$

Apart from the usual complex conjugation there is the following conjugation for harmonics [25],

$$\widetilde{u_i^{(\pm,0)}} = u^{(0,\pm)i}, \quad \widetilde{u_i^{(0,\pm)}} = u^{(\pm,0)i}, \quad \widetilde{u^{(\pm,0)i}} = -u_i^{(0,\pm)}, \quad \widetilde{u^{(0,\pm)i}} = -u_i^{(\pm,0)}. \quad (2.20)$$

It is the conjugation (2.20) which allows one to define real objects in harmonic superspace with $USp(4)$ harmonics.

The invariant Cartan forms on the $USp(4)$ group are given by

$$\begin{aligned} \omega^{(\pm\pm,0)} &= u^{(\pm,0)i} du_i^{(\pm,0)}, & \omega^{(0,\pm\pm)} &= u^{(0,\pm)i} du_i^{(0,\pm)}, \\ \omega^{(\pm,\pm)} &= \frac{1}{2}(u^{(\pm,0)i} du_i^{(0,\pm)} + u^{(0,\pm)i} du_i^{(\mp,0)}), \\ \omega^{(\pm,\mp)} &= \frac{1}{2}(u^{(0,\mp)i} du_i^{(\pm,0)} + u^{(\pm,0)i} du_i^{(0,\mp)}), \\ \omega_1^{(0,0)} &= \frac{1}{2}(u^{(+,0)i} du_i^{(-,0)} - u_i^{(-,0)} du^{(+,0)i}), \\ \omega_2^{(0,0)} &= \frac{1}{2}(u^{(0,+)i} du_i^{(0,-)} - u_i^{(0,-)} du^{(0,+)i}). \end{aligned} \quad (2.21)$$

These differential forms will be further used in the construction of the superparticle Lagrangian on $\mathcal{N}=4$ harmonic superspace with $USp(4)$ harmonic variables.

3 $\mathcal{N}=4$ harmonic superparticle model

In this section we construct the $\mathcal{N}=4$ harmonic superparticle model extending the conventional $\mathcal{N}=4$ superparticle by the Lagrangian for the harmonic variables on the coset $USp(4)/(U(1) \times U(1))$. Then we develop the Hamiltonian formulation and analyze the constraints. We assume everywhere that any Lagrangian L defines an action S by the rule $S = \int d\tau L$, where τ is worldline parameter. The derivatives of all superspace variables over τ are denoted by dots, e.g., $\dot{x}^m = dx^m/d\tau$, $\dot{\theta}_{i\alpha} = d\theta_{i\alpha}/d\tau$, etc.

3.1 $\mathcal{N}=4$ superparticle Lagrangian

The Lagrangian for a massive superparticle moving in $\mathcal{N}=4$ superspace can be written in terms of Cartan forms (2.7) in a standard way [14, 15, 16, 17],⁴

$$L_{\mathcal{N}=4} = L_1 + L_2, \quad (3.1)$$

$$L_1 = -\frac{1}{2}(e^{-1}\dot{\omega}^m \dot{\omega}_m + em^2), \quad (3.2)$$

$$L_2 = -(Z^{ij}\theta_i^\alpha \dot{\theta}_{j\alpha} + \bar{Z}_{ij}\bar{\theta}_{\dot{\alpha}}^i \dot{\bar{\theta}}^{j\dot{\alpha}}). \quad (3.3)$$

⁴Here we assume that Cartan forms ω^M are pulled back $\omega^M = \dot{\omega}^M(\tau)d\tau$ on the superparticle worldline $Z^M = Z^M(\tau)$, where Z^M is a set of superspace coordinates. We hope that our notations for the Cartan forms ω^M and their values $\dot{\omega}^M$ do not lead to misunderstandings. The same concerns the Cartan forms for harmonics (2.21) and their values appearing in the Lagrangian (3.20).

Here L_1 is a kinetic term with the mass m and L_2 is the Wess-Zumino term with the central charges Z^{ij} , \bar{Z}_{ij} . Further we will consider only the case when the central charges are given by (2.5), so the Lagrangian L_2 takes the following form

$$L_2 = -(z\Omega^{ij}\theta_i^\alpha\dot{\theta}_{j\alpha} - \bar{z}\Omega_{ij}\bar{\theta}_\alpha^i\dot{\bar{\theta}}^{j\dot{\alpha}}). \quad (3.4)$$

The superparticle action is invariant under supertranslations,

$$\begin{aligned} \delta_\epsilon \theta_i^\alpha &= \epsilon_i^\alpha, & \delta_\epsilon \bar{\theta}^{i\dot{\alpha}} &= \bar{\epsilon}^{i\dot{\alpha}}, \\ \delta_\epsilon x^m &= -i\epsilon_i \sigma^m \bar{\theta}^i + i\theta_i \sigma^m \bar{\epsilon}^i, \end{aligned} \quad (3.5)$$

which lead to the conserved charges (supercharges),

$$Q_\alpha^i = 2ie^{-1}\dot{\omega}_m(\sigma^m \bar{\theta}^i)_\alpha + 2z\Omega^{ij}\theta_{j\alpha}, \quad \bar{Q}_{i\dot{\alpha}} = -2ie^{-1}\dot{\omega}_m(\theta_i \sigma^m)_{\dot{\alpha}} - 2\bar{z}\Omega_{ij}\bar{\theta}_{\dot{\alpha}}^j. \quad (3.6)$$

Upon quantization, the supercharges (3.6) turn into the differential operators (2.1) with the superalgebra (2.2).

If the BPS condition (2.4) is satisfied, the Lagrangian (3.1) respects also κ -symmetry,

$$\begin{aligned} \delta_\kappa \theta_{i\alpha} &= -ip_m(\sigma^m \bar{\kappa}_i)_\alpha - \bar{z}\Omega_{ij}\kappa_\alpha^j, & \delta_\kappa \bar{\theta}_\alpha^i &= ip_m(\kappa^i \sigma^m)_{\dot{\alpha}} + z\Omega^{ij}\bar{\kappa}_{j\dot{\alpha}}, \\ \delta_\kappa x^m &= i\delta_\kappa \theta_i \sigma^m \bar{\theta}^i - i\theta_i \sigma^m \delta_\kappa \bar{\theta}^i, & \delta_\kappa e &= -4(\bar{\kappa}_{i\dot{\alpha}}\dot{\bar{\theta}}^{i\dot{\alpha}} + \dot{\theta}_i^\alpha \kappa_\alpha^i), \end{aligned} \quad (3.7)$$

where κ_α^i , $\bar{\kappa}_{i\dot{\alpha}}$ are anticommuting local parameters. Despite the relation (2.4) defines the central charge z only up to a phase, we fix this freedom without loss of generality as

$$m = -iz = i\bar{z}. \quad (3.8)$$

As it will be shown further, it is the relation (3.8) that provides us with the correct Dirac equations for physical spinors in massive supermultiplets.

3.2 Momenta, constraints and Hamiltonian

We introduce the canonical momenta for the superspace variables as follows,

$$p_m = -\frac{\partial L_{\mathcal{N}=4}}{\partial \dot{x}^m} = e^{-1}\dot{\omega}^m, \quad (3.9)$$

$$\pi_\alpha^i = \frac{\partial L_{\mathcal{N}=4}}{\partial \dot{\theta}_i^\alpha} = ip_m(\sigma^m \bar{\theta}^i)_\alpha + z\Omega^{ij}\theta_{j\alpha}, \quad (3.10)$$

$$\bar{\pi}_{i\dot{\alpha}} = \frac{\partial L_{\mathcal{N}=4}}{\partial \dot{\bar{\theta}}^{i\dot{\alpha}}} = ip_m(\theta_i \sigma^m)_{\dot{\alpha}} + \bar{z}\Omega_{ij}\bar{\theta}_{\dot{\alpha}}^j = -(\pi_\alpha^i)^*. \quad (3.11)$$

The spinorial momenta (3.10,3.11) do not allow one to express the corresponding velocities and therefore they are considered as the constraints,

$$\begin{aligned} D_\alpha^i &= -\pi_\alpha^i + ip_m(\sigma^m \bar{\theta}^i)_\alpha + z\Omega^{ij}\theta_{j\alpha} \approx 0, \\ \bar{D}_{i\dot{\alpha}} &= \bar{\pi}_{i\dot{\alpha}} - ip_m(\theta_i \sigma^m)_{\dot{\alpha}} - \bar{z}\Omega_{ij}\bar{\theta}_{\dot{\alpha}}^j \approx 0. \end{aligned} \quad (3.12)$$

As usual, the first and second class constraints are entangled in (3.12) and their separation can be done, e.g., with the help of space-time harmonic variables [30, 31]. However, we do not need this technique here since the correct accounting of these constraints will be done in the next section by applying Gupta-Bleuler quantization method in harmonic superspace.

To get the Hamiltonian for $\mathcal{N}=4$ superparticle we perform the Legendre transform,

$$H_{\mathcal{N}=4} = -\dot{x}^m p_m + \dot{\theta}_i^\alpha \pi_\alpha^i + \dot{\bar{\theta}}_{\dot{\alpha}}^i \bar{\pi}_i^{\dot{\alpha}} - L_{\mathcal{N}=4} = -\frac{e}{2}(p^2 - m^2). \quad (3.13)$$

The Hamiltonian is proportional to the first-class mass-shell constraint

$$p^2 - m^2 \approx 0 \quad (3.14)$$

with the Lagrange multiplier e .

The Poisson brackets are defined in a standard way,

$$[x^m, p_n]_P = -\delta_n^m, \quad \{\theta_i^\alpha, \pi_\beta^j\}_P = -\delta_i^j \delta_\beta^\alpha, \quad \{\bar{\theta}^{i\dot{\alpha}}, \bar{\pi}_{j\dot{\beta}}\}_P = -\delta_j^i \delta_{\dot{\beta}}^{\dot{\alpha}}. \quad (3.15)$$

With the help of the Poisson brackets we write down the variation for κ -symmetry transformations,

$$\delta_\kappa = \kappa^{i\alpha} [\psi_{i\alpha}, \cdot]_P + \bar{\kappa}_{i\dot{\alpha}} [\bar{\psi}^{i\dot{\alpha}}, \cdot]_P, \quad (3.16)$$

where

$$\psi_{i\alpha} = -ip_m \sigma_\alpha^m \bar{D}_{i\dot{\alpha}} + \bar{z} \Omega_{ik} D_\alpha^k, \quad \bar{\psi}^{i\dot{\alpha}} = ip_m \sigma^{m\alpha\dot{\alpha}} D_\alpha^i - z \Omega^{ik} \bar{D}_{\dot{\alpha}k} \quad (3.17)$$

are the generators of κ -transformations (3.7).

3.3 Lagrangian for $\mathcal{N}=4$ harmonic superparticle

Let us consider now the $\mathcal{N}=4$ harmonic superspace with $USp(4)$ harmonics, $Z_H = \{x^m, \theta_{i\alpha}, \bar{\theta}_{\dot{\alpha}}^i, u\}$, where the harmonics u are $USp(4)$ matrices defined in (2.8). The superparticle Lagrangian (3.1) should be supplemented by the harmonic term,

$$L = L_{\mathcal{N}=4} + L_{USp(4)}, \quad (3.18)$$

where

$$L_{USp(4)} = L_\omega + L_{WZ} + L_\lambda, \quad (3.19)$$

$$L_\omega = \frac{2R^2}{e} [\dot{\omega}^{(++,0)} \dot{\omega}^{(-,0)} + \dot{\omega}^{(0,++)} \dot{\omega}^{(0,-)} + \dot{\omega}^{(+,+)} \dot{\omega}^{(-,-)} + \dot{\omega}^{(+,-)} \dot{\omega}^{(-,+)}], \quad (3.20)$$

$$L_{WZ} = -\frac{is_1}{2} [u^{(+,0)i} \dot{u}_i^{(-,0)} - \dot{u}^{(+,0)i} u_i^{(-,0)}] - \frac{is_2}{2} [u^{(0,+)i} \dot{u}_i^{(0,-)} - \dot{u}^{(0,+)i} u_i^{(0,-)}], \quad (3.21)$$

$$L_\lambda = \lambda_1 (u_i^{(+,0)} u^{(-,0)i} - 1) + \lambda_2 (u_i^{(0,+)} u^{(0,-)i} - 1) + \lambda^{(-,-)} u_i^{(+,0)} u^{(0,+)i} + \lambda^{(-,+)} u_i^{(+,0)} u^{(0,-)i} + \lambda^{(+,-)} u_i^{(0,+)} u^{(-,0)i} + \lambda^{(+,+)} u_i^{(-,0)} u^{(0,-)i}. \quad (3.22)$$

Here L_ω is built out from the Cartan forms (2.21) and describes the kinetic term for harmonics, L_{WZ} is the Wess-Zumino term and L_λ takes into account the constraints (2.17, 2.18) with the Lagrange multipliers $\lambda_1, \lambda_2, \lambda^{(+,+)}, \lambda^{(-,-)}, \lambda^{(-,+)}, \lambda^{(+,-)}$. R, s_1, s_2 are some constants.

The main advantage of using the harmonic superspace Z_H is the possibility of passing to the harmonic projections for all objects with $USp(4)$ indices. For instance, for the Grassmann variables we have

$$\theta_\alpha^I = -u^I \theta_{i\alpha}, \quad \bar{\theta}_{\dot{\alpha}}^I = u^I \bar{\theta}_{\dot{\alpha}}^i, \quad (3.23)$$

where the index I takes the following values

$$I = \{(+, 0), (-, 0), (0, +), (0, -)\}. \quad (3.24)$$

One can promote the conjugation (2.20) to such objects,

$$\widetilde{\theta_\alpha^{(\pm,0)}} = \bar{\theta}_{\dot{\alpha}}^{(0,\pm)}, \quad \widetilde{\theta_\alpha^{(0,\pm)}} = \bar{\theta}_{\dot{\alpha}}^{(\pm,0)}, \quad \widetilde{\bar{\theta}_{\dot{\alpha}}^{(0,\pm)}} = -\theta_\alpha^{(\pm,0)}, \quad \widetilde{\bar{\theta}_{\dot{\alpha}}^{(\pm,0)}} = -\theta_\alpha^{(0,\pm)}. \quad (3.25)$$

Analogously, we project the constraints (3.12) with harmonics,

$$\begin{aligned} D_\alpha^I &= u^I_i D_\alpha^i = -u^I_i \pi_\alpha^i + ip_m (\sigma^m \bar{\theta}^I)_\alpha - z \theta_\alpha^I \approx 0, \\ \bar{D}_{\dot{\alpha}}^I &= -u^{Ii} \bar{D}_{i\dot{\alpha}} = -u^{Ii} \bar{\pi}_{i\dot{\alpha}} - ip_m (\theta^I \sigma^m)_{\dot{\alpha}} + \bar{z} \bar{\theta}_{\dot{\alpha}}^I \approx 0. \end{aligned} \quad (3.26)$$

They are also related by the conjugation as

$$\widetilde{D_\alpha^{(\pm,0)}} = -\bar{D}_{\dot{\alpha}}^{(0,\pm)}, \quad \widetilde{D_\alpha^{(0,\pm)}} = -\bar{D}_{\dot{\alpha}}^{(\pm,0)}, \quad \widetilde{\bar{D}_{\dot{\alpha}}^{(\pm,0)}} = D_\alpha^{(0,\pm)}, \quad \widetilde{\bar{D}_{\dot{\alpha}}^{(0,\pm)}} = D_\alpha^{(\pm,0)}. \quad (3.27)$$

Let us give here also the harmonic projections of the κ -symmetry constraints (3.17),

$$\psi_\alpha^I = -ip_m \sigma_{\alpha\dot{\alpha}}^m \bar{D}^{I\dot{\alpha}} + \bar{z} D_\alpha^I \approx 0, \quad \bar{\psi}_{\dot{\alpha}}^I = -ip_m \sigma_{\alpha\dot{\alpha}}^m D^{I\alpha} + z \bar{D}_{\dot{\alpha}}^I \approx 0, \quad (3.28)$$

where index I takes the values (3.24). However, apart from (3.26) and (3.28) there are harmonic constraints originating from the symmetries of the Lagrangian for harmonics (3.19). We analyze them in details in the next subsection.

3.4 Constraints and Hamiltonian for harmonic variables

Let us define the canonical momenta for harmonic variables,

$$\begin{aligned} v^{(\pm,0)i} &= -\frac{\partial L_{USp(4)}}{\partial \dot{u}_i^{(\mp,0)}} = -\frac{R^2}{e} [2u^{(\mp,0)i} \dot{\omega}^{(\pm\pm,0)} + u^{(0,\mp)i} \dot{\omega}^{(\pm,\pm)} + u^{(0,\pm)i} \dot{\omega}^{(\pm,\mp)}] - \frac{is_1}{2} u^{(\pm,0)i}, \\ v^{(0,\pm)i} &= -\frac{\partial L_{USp(4)}}{\partial \dot{u}_i^{(0,\mp)}} = -\frac{R^2}{e} [u^{(\mp,0)i} \dot{\omega}^{(\pm,\pm)} + u^{(\pm,0)i} \dot{\omega}^{(\mp,\pm)} + 2u^{(0,\mp)i} \dot{\omega}^{(0,\pm\pm)}] - \frac{is_2}{2} u^{(0,\pm)i}. \end{aligned} \quad (3.29)$$

They are used in constructing of covariant harmonic momenta,

$$S_1 = u_i^{(+,0)} v^{(-,0)i} - u_i^{(-,0)} v^{(+,0)i} = -is_1, \quad (3.30a)$$

$$S_2 = u_i^{(0,+)} v^{(0,-)i} - u_i^{(0,-)} v^{(0,+)i} = -is_2, \quad (3.30b)$$

$$D^{(\pm\pm,0)} = u_i^{(\pm,0)} v^{(\pm,0)i} = \mp \frac{2R^2}{e} \dot{\omega}^{(\pm\pm,0)}, \quad (3.30c)$$

$$D^{(0,\pm\pm)} = u_i^{(0,\pm)} v^{(0,\pm)i} = \mp \frac{2R^2}{e} \dot{\omega}^{(0,\pm\pm)}, \quad (3.30d)$$

$$D^{(\pm,\pm)} = u_i^{(\pm,0)} v^{(0,\pm)i} + u_i^{(0,\pm)} v^{(\pm,0)i} = \mp \frac{2R^2}{e} \dot{\omega}^{(\pm,\pm)}, \quad (3.30e)$$

$$D^{(\pm,\mp)} = u_i^{(\pm,0)} v^{(0,\mp)i} - u_i^{(0,\mp)} v^{(\pm,0)i} = \mp \frac{2R^2}{e} \dot{\omega}^{(\pm,\mp)}. \quad (3.30f)$$

There are six constraints with the canonical momenta (3.29)

$$\begin{aligned} C_1 &= u_i^{(+,0)} v^{(-,0)i} + u_i^{(-,0)} v^{(+,0)i} = 0, & C_2 &= u_i^{(0,+)} v^{(0,-)i} + u_i^{(0,-)} v^{(0,+)i} = 0, \\ C^{(\pm,\pm)} &= u_i^{(\pm,0)} v^{(0,\pm)i} - u_i^{(0,\pm)} v^{(\pm,0)i} = 0, & C^{(\pm,\mp)} &= u_i^{(\pm,0)} v^{(0,\mp)i} + u_i^{(0,\mp)} v^{(\pm,0)i} = 0. \end{aligned} \quad (3.31)$$

Equations (3.30,3.31) considered together allow one to express all sixteen harmonic momenta (3.29) through the variables S , D , C in the lhs in (3.30,3.31). Therefore we will use further the covariant momenta (3.30) instead of canonical ones.

There are also the following constraints for the harmonic variables which appear by varying (3.19) over Lagrange multipliers,

$$\begin{aligned} \chi_1 &= \frac{\partial L_{USp(4)}}{\partial \lambda_1} = u_i^{(+,0)} u^{(-,0)i} - 1 = 0, & \chi_2 &= \frac{\partial L_{USp(4)}}{\partial \lambda_2} = u_i^{(0,+)} u^{(0,-)i} - 1 = 0, \\ \chi^{(\pm,\pm)} &= \frac{\partial L_{USp(4)}}{\partial \lambda^{(\mp,\mp)}} = u_i^{(\pm,0)} u^{(0,\pm)i} = 0, & \chi^{(\pm,\mp)} &= \frac{\partial L_{USp(4)}}{\partial \lambda^{(\mp,\pm)}} = u_i^{(\pm,0)} u^{(0,\mp)i} = 0. \end{aligned} \quad (3.32)$$

Let us introduce the Poisson brackets for the harmonic variables and corresponding momenta,

$$[u_i^{(\pm,0)}, v^{(\mp,0)j}]_P = -\delta_i^j, \quad [u_i^{(0,\pm)}, v^{(0,\mp)j}]_P = -\delta_i^j, \quad \text{other brackets vanish.} \quad (3.33)$$

It is easy to see that the functions (3.30a,3.30b) commute weakly under these brackets with the constraints (3.31,3.32) and therefore they belong to the first class according to the Dirac's terminology. Another first-class constraint appears as the equation of motion for the einbein field,

$$\begin{aligned} 0 = \frac{\partial L}{\partial e} &= \frac{1}{2} (p^m p_m - m^2 + D^{(++,0)} D^{(--,0)} + D^{(0,++)} D^{(0,--)} \\ &\quad + D^{(+,+)} D^{(-,-)} + D^{(+,-)} D^{(-,+)}). \end{aligned} \quad (3.34)$$

This equation is a modification of the mass-shell constraint (3.14) with the covariant momenta for the harmonic variables.

Let us now turn to the second-class constraints for the harmonic variables. We denote the covariant momenta (3.30,3.31) and constraints (3.32) as

$$\begin{aligned} D^M &= \{S_1, S_2, D^{(++,0)}, D^{(--,0)}, D^{(0,++)}, D^{(0,--)}, D^{(+,+)}, D^{(-,-)}, D^{(+,-)}, D^{(-,+)}\}, \\ C^I &= \{C_1, C_2, C^{(+,+)}, C^{(-,-)}, C^{(+,-)}, C^{(-,+)}\}, \\ \chi_I &= \{\chi_1, \chi_2, -\chi^{(-,-)}, \chi^{(+,+)}, -\chi^{(-,+)}, \chi^{(+,-)}\} \end{aligned} \quad (3.35)$$

and calculate their Poisson brackets,

$$[C^I, \chi_J]_P = 2\delta_J^I, \quad [C^I, C^J] = F_M^{IJ} D^M, \quad (3.36)$$

where F_M^{IJ} is some constant matrix. Equations (3.36) show that the constraints (3.31,3.32) are second-class and therefore should be taken into account with the help of the Dirac bracket,

$$\begin{aligned} [A, B]_D &= [A, B]_P + \frac{1}{2}[A, C^I]_P[\chi_I, B]_P - \frac{1}{2}[A, \chi_I]_P[C^I, B]_P \\ &\quad - \frac{1}{4}[A, \chi_I]_P F_M^{IJ} D^M [\chi_J, B]_P, \end{aligned} \quad (3.37)$$

where A, B are arbitrary two functions on a phase space for harmonic variables.

Now we are ready to define the Hamiltonian for the particle on $USp(4)/(U(1) \times U(1))$ coset as a Legendre transform for the Lagrangian (3.19),

$$H_{USp(4)} = -\dot{u}_i^{(+,0)} v^{(-,0)i} - \dot{u}_i^{(-,0)} v^{(+,0)i} - \dot{u}_i^{(0,+)} v^{(0,-)i} - \dot{u}_i^{(0,-)} v^{(0,+)i} - L_{USp(4)}. \quad (3.38)$$

Expressing the velocities for harmonic variables from (3.29) through the momenta and substituting them into (3.38) we find

$$\begin{aligned} H_{USp(4)} &= 2R^2 e^{-1} [\dot{\omega}^{(++,0)} \dot{\omega}^{(--,0)} + \dot{\omega}^{(0,++)} \dot{\omega}^{(0,--)} + \dot{\omega}^{(+,+)} \dot{\omega}^{(-,-)} + \dot{\omega}^{(+,-)} \dot{\omega}^{(-,+)}] \\ &\quad - \omega_1^{(0,0)} (S_1 - is_1) - \omega_2^{(0,0)} (S_2 - is_2) - L_\lambda, \end{aligned} \quad (3.39)$$

where L_λ is given by (3.22). Since the constraints (3.32) are accounted by the Dirac bracket (3.37), we omit L_λ further. Note that the functions $\omega_1^{(0,0)}$, $\omega_2^{(0,0)}$ are arbitrary, we treat them as the Lagrange multipliers and denote further as μ , ν , respectively. As a result, the Hamiltonian (3.39) is given by

$$\begin{aligned} H_{USp(4)} &= -\frac{e}{2R^2} [D^{(++,0)} D^{(--,0)} + D^{(0,++)} D^{(0,--)} + D^{(+,+)} D^{(-,-)} + D^{(+,-)} D^{(-,+)}] \\ &\quad - \mu (S_1 - is_1) - \nu (S_2 - is_2). \end{aligned} \quad (3.40)$$

The Hamiltonian describing the dynamics of both superspace and harmonic variables reads

$$H = H_{USp(4)} + H_{\mathcal{N}=4}, \quad (3.41)$$

where $H_{\mathcal{N}=4}$ and $H_{USp(4)}$ are given by (3.13) and (3.40), respectively.

4 Gupta-Bleuler quantization of massive harmonic superparticle with central charge term

According to the canonical quantization, one replaces the canonical momenta (3.9–3.11), (3.29) with the following differential operators,

$$p_m \rightarrow i\partial_m, \quad \pi_\alpha^i \rightarrow -i\frac{\partial}{\partial\theta_i^\alpha}, \quad \bar{\pi}_{i\dot{\alpha}} = -i\frac{\partial}{\partial\bar{\theta}^{i\dot{\alpha}}}, \quad v^{(\pm,0)i} \rightarrow \frac{\partial}{\partial u_i^{(\mp,0)}}, \quad v^{(0,\pm)i} \rightarrow \frac{\partial}{\partial u_i^{(0,\mp)}}. \quad (4.1)$$

The covariant harmonic momenta (3.30) turn into the harmonic derivatives (2.16), while the Grassmann constraints (3.26) correspond to the following covariant spinor derivatives⁵

$$\begin{aligned} D_\alpha^{(\pm,0)} &= \pm \frac{\partial}{\partial\theta^{(\mp,0)\alpha}} + i(\sigma^m \bar{\theta}^{(\pm,0)})_\alpha \partial_m + iz\theta_\alpha^{(\pm,0)}, \\ D_\alpha^{(0,\pm)} &= \pm \frac{\partial}{\partial\theta^{(0,\mp)\alpha}} + i(\sigma^m \bar{\theta}^{(0,\pm)})_\alpha \partial_m + iz\theta_\alpha^{(0,\pm)}, \\ \bar{D}_{\dot{\alpha}}^{(\pm,0)} &= \pm \frac{\partial}{\partial\bar{\theta}^{(\mp,0)\dot{\alpha}}} - i(\theta^{(\pm,0)}\sigma^m)_{\dot{\alpha}} \partial_m - i\bar{z}\bar{\theta}_{\dot{\alpha}}^{(\pm,0)}, \\ \bar{D}_{\dot{\alpha}}^{(0,\pm)} &= \pm \frac{\partial}{\partial\bar{\theta}^{(0,\mp)\dot{\alpha}}} - i(\theta^{(0,\mp)}\sigma^m)_{\dot{\alpha}} \partial_m - i\bar{z}\bar{\theta}_{\dot{\alpha}}^{(0,\pm)}. \end{aligned} \quad (4.2)$$

with non-trivial anticommutation relations given by

$$\begin{aligned} \{D_\alpha^{(+,0)}, D_\beta^{(-,0)}\} &= \{D_\alpha^{(0,+)}, D_\beta^{(0,-)}\} = -2iz\varepsilon_{\alpha\beta}, \\ \{\bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\beta}}^{(-,0)}\} &= \{\bar{D}_{\dot{\alpha}}^{(0,+)}, \bar{D}_{\dot{\beta}}^{(0,-)}\} = 2i\bar{z}\varepsilon_{\dot{\alpha}\dot{\beta}}, \\ \{D_\alpha^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(-,0)}\} &= \{D_\alpha^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\} = -2i\sigma_{\alpha\dot{\alpha}}^m \partial_m. \end{aligned} \quad (4.3)$$

The operators (4.1) should be realized in some Hilbert space formed by the superfunctions

$$\Phi = \Phi(x^m, \theta_{i\alpha}, \bar{\theta}_{\dot{\alpha}}^i, u), \quad (4.4)$$

which should satisfy some equations of motion and constraints originating from the superparticle constraints. The superparticle has both first- and second-class constraints. The first-class constraints (3.30a, 3.30b, 3.34) form closed algebra under the Poisson or Dirac bracket. Therefore, they all should be imposed on state vectors,⁶

$$S_1\Phi^{(s_1, s_2)} = s_1\Phi^{(s_1, s_2)}, \quad S_2\Phi^{(s_1, s_2)} = s_2\Phi^{(s_1, s_2)}, \quad (4.5)$$

$$[\partial^m \partial_m - \frac{1}{R^2}X + m^2]\Phi^{(s_1, s_2)} = 0, \quad (4.6)$$

⁵The operators $D_\alpha^i, \bar{D}_{i\dot{\alpha}}$ are multiplied here by $-i$ for convenience.

⁶Further we label the functions Φ by the values of $U(1)$ charges as $\Phi^{(s_1, s_2)}$.

where ⁷

$$X = D^{(-,0)}D^{(+,0)} + D^{(0,-)}D^{(0,++)} + D^{(-,-)}D^{(+,+)} + D^{(+,-)}D^{(-,+)}. \quad (4.7)$$

The second-class constraints should be accounted either by constructing the corresponding Dirac bracket or by applying Gupta-Bleuler method. In our case the second-class harmonic constraints (3.31,3.32) are taken into account by the Dirac bracket (3.37), while the spinorial ones (3.26) should be accounted à la Gupta-Bleuler. It means that they have to be divided into two complex conjugate subsets with weakly commuting constraints in each subset. As follows from the algebra (4.3), there are two ways of separating the derivatives (4.2) into such subsets:

$$\{D_\alpha^{(+,0)}, D_\alpha^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}\} \cup \{D_\alpha^{(-,0)}, D_\alpha^{(0,-)}, \bar{D}_{\dot{\alpha}}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\}, \quad (4.8)$$

$$\{D_\alpha^{(+,0)}, D_\alpha^{(0,-)}, \bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\} \cup \{D_\alpha^{(-,0)}, D_\alpha^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}\}. \quad (4.9)$$

Both these lines (4.8,4.9) lead to the equivalent superfield realizations of supersymmetry. Therefore, we consider in details only (4.8) and give short comments on the second case (4.9) in the end of this section on a particular example. Therefore we require the superfield $\Phi^{(s_1,s_2)}$ to be analytic,

$$D_\alpha^{(+,0)}\Phi^{(s_1,s_2)} = D_\alpha^{(0,+)}\Phi^{(s_1,s_2)} = \bar{D}_{\dot{\alpha}}^{(+,0)}\Phi^{(s_1,s_2)} = \bar{D}_{\dot{\alpha}}^{(0,+)}\Phi^{(s_1,s_2)} = 0. \quad (4.10)$$

Note that the physical states should respect the first-class κ -symmetry constraints (3.28) which are not accounted so far. The problem is that the harmonic part of the superparticle Lagrangian (3.19) violates the κ -symmetry (3.7). As follows from (4.6), the states $\Phi^{(s_1,s_2)}$ acquire additional masses due to the eigenvalues of the operator X and the BPS condition (2.4) is violated. This is not surprising as we consider a superparticle in the harmonic superspace $Z_H = \{x^m, \theta_{i\alpha}, \bar{\theta}_{\dot{\alpha}}^i, u\}$, where the harmonic variables have non-trivial dynamics rather than playing auxiliary role. To resolve this problem and to obtain the physical states describing irreducible representations of supersymmetry algebra we have to “freeze” the harmonic dynamics by imposing additional harmonic constraints on the superfield $\Phi^{(s_1,s_2)}$. Such constraints should be first-class and should be compatible with the second-class constraints (4.10). Since the harmonic derivatives $D^{(++,0)}$, $D^{(0,++)}$, $D^{(+,+)}$, $D^{(-,+)}$, $D^{(+,-)}$ leave the set of Grassmann derivatives in (4.10) invariant, we require them to annihilate the state,

$$D^{(++,0)}\Phi^{(s_1,s_2)} = D^{(0,++)}\Phi^{(s_1,s_2)} = D^{(+,+)}\Phi^{(s_1,s_2)} = D^{(-,+)}\Phi^{(s_1,s_2)} = D^{(+,-)}\Phi^{(s_1,s_2)} = 0. \quad (4.11)$$

It is easy to see that the state under constraints (4.11) belongs to the kernel of the operator X , i.e. $X\Phi^{(s_1,s_2)} = 0$. Therefore the mass-shell constraint (4.6) reduces to

$$(\partial^m \partial_m + m^2)\Phi^{(s_1,s_2)} = 0, \quad (4.12)$$

⁷Here we use a particular ordering of the harmonic derivatives although other orderings are also possible.

that is nothing but the usual Klein-Gordon equation. The BPS condition (2.4) is now restored and the state respects the constraints of κ -symmetry which eliminate the unphysical degrees of freedom.

Upon quantization, the generators of κ -symmetry (3.28) turn into the differential operators,

$$\psi_\alpha^I = \sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{D}^{I\dot{\alpha}} + \bar{z} D_\alpha^I, \quad \bar{\psi}_{\dot{\alpha}}^I = \sigma_{\alpha\dot{\alpha}}^m \partial_m D^{I\alpha} + z \bar{D}_{\dot{\alpha}}^I. \quad (4.13)$$

Owing to the analyticity (4.10), we have to impose only the following constraints,

$$\begin{aligned} (\sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{D}^{(-,0)\dot{\alpha}} + \bar{z} D_\alpha^{(-,0)}) \Phi^{(s_1, s_2)} &= 0, \\ (\sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{D}^{(0,-)\dot{\alpha}} + \bar{z} D_\alpha^{(0,-)}) \Phi^{(s_1, s_2)} &= 0, \\ (\sigma_{\alpha\dot{\alpha}}^m \partial_m D^{(-,0)\alpha} + z \bar{D}_{\dot{\alpha}}^{(-,0)}) \Phi^{(s_1, s_2)} &= 0, \\ (\sigma_{\alpha\dot{\alpha}}^m \partial_m D^{(0,-)\alpha} + z \bar{D}_{\dot{\alpha}}^{(0,-)}) \Phi^{(s_1, s_2)} &= 0. \end{aligned} \quad (4.14)$$

The relations (4.14) can be brought to more useful form. For this purpose we introduce the operators

$$\begin{aligned} Y^{(--,0)} &= \frac{i}{4} (z \bar{D}_{\dot{\alpha}}^{(-,0)} \bar{D}^{(-,0)\dot{\alpha}} - \bar{z} D^{(-,0)\alpha} D_\alpha^{(-,0)}), \\ Y^{(0,--)} &= \frac{i}{4} (z \bar{D}_{\dot{\alpha}}^{(0,-)} \bar{D}^{(0,-)\dot{\alpha}} - \bar{z} D^{(0,-)\alpha} D_\alpha^{(0,-)}), \end{aligned} \quad (4.15)$$

which have the following commutators with (4.13)

$$\begin{aligned} \frac{1}{z} [D_\alpha^{(+,0)}, Y^{(--,0)}] &= \sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{D}^{(-,0)\dot{\alpha}} + \bar{z} D_\alpha^{(-,0)}, \\ \frac{1}{z} [D_\alpha^{(0,+)}, Y^{(0,--)}] &= \sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{D}^{(0,-)\dot{\alpha}} + \bar{z} D_\alpha^{(0,-)}, \\ -\frac{1}{\bar{z}} [\bar{D}_{\dot{\alpha}}^{(+,0)}, Y^{(--,0)}] &= \sigma_{\alpha\dot{\alpha}}^m \partial_m D^{(-,0)\alpha} + z \bar{D}_{\dot{\alpha}}^{(-,0)}, \\ -\frac{1}{\bar{z}} [\bar{D}_{\dot{\alpha}}^{(0,+)}, Y^{(0,--)}] &= \sigma_{\alpha\dot{\alpha}}^m \partial_m D^{(0,-)\alpha} + z \bar{D}_{\dot{\alpha}}^{(0,-)}. \end{aligned} \quad (4.16)$$

Therefore instead of (4.14) we impose the following first-class constraints on the superfield,

$$\begin{aligned} (z \bar{D}_{\dot{\alpha}}^{(-,0)} \bar{D}^{(-,0)\dot{\alpha}} - \bar{z} D^{(-,0)\alpha} D_\alpha^{(-,0)}) \Phi^{(s_1, s_2)} &= 0, \\ (z \bar{D}_{\dot{\alpha}}^{(0,-)} \bar{D}^{(0,-)\dot{\alpha}} - \bar{z} D^{(0,-)\alpha} D_\alpha^{(0,-)}) \Phi^{(s_1, s_2)} &= 0. \end{aligned} \quad (4.17)$$

In fact, the constraints (4.17) are stronger than (4.14). Nevertheless, as is argued in [18, 19] for $\mathcal{N}=2$ superparticle, such first-class constraints should be imposed on the state since they appear as functions of second-class constraints (3.26). Moreover, they can be considered as the generators of symmetries of the superparticle Lagrangian as is shown in the Appendix 2 for the massless case.

Let us summarize all the equations for the superfield $\Phi^{(s_1, s_2)}$ in a single list

$$\begin{aligned}
S_1 \Phi^{(s_1, s_2)} &= s_1 \Phi^{(s_1, s_2)}, & S_2 \Phi^{(s_1, s_2)} &= s_2 \Phi^{(s_1, s_2)}, \\
D_\alpha^{(+,0)} \Phi^{(s_1, s_2)} &= D_\alpha^{(0,+)} \Phi^{(s_1, s_2)} = \bar{D}_{\dot{\alpha}}^{(+,0)} \Phi^{(s_1, s_2)} = \bar{D}_{\dot{\alpha}}^{(0,+)} \Phi^{(s_1, s_2)} = 0, \\
D^{(++,0)} \Phi^{(s_1, s_2)} &= D^{(0,++)} \Phi^{(s_1, s_2)} = D^{(+,+)} \Phi^{(s_1, s_2)} = 0, \\
D^{(-,+)} \Phi^{(s_1, s_2)} &= D^{(+,-)} \Phi^{(s_1, s_2)} = 0, \\
(z \bar{D}_{\dot{\alpha}}^{(-,0)} \bar{D}^{(-,0)\dot{\alpha}} - \bar{z} D^{(-,0)\alpha} D_\alpha^{(-,0)}) \Phi^{(s_1, s_2)} &= 0, \\
(z \bar{D}_{\dot{\alpha}}^{(0,-)} \bar{D}^{(0,-)\dot{\alpha}} - \bar{z} D^{(0,-)\alpha} D_\alpha^{(0,-)}) \Phi^{(s_1, s_2)} &= 0, \\
(\partial^m \partial_m + m^2) \Phi^{(s_1, s_2)} &= 0.
\end{aligned} \tag{4.18}$$

Note that due to the algebra (A.3) the derivatives $D^{(+,-)}$, $D^{(-,+)}$ commute as $[D^{(+,-)}, D^{(-,+)}] = S_2 - S_1$. Hence, the operator $S_2 - S_1$ also annihilates the state, $(S_2 - S_1) \Phi^{(s_1, s_2)} = 0$ and the resulting superfield has equal $U(1)$ charges, $s_1 = s_2$.

We point out that the constraints $D^{(-,+)} \Phi^{(s_1, s_2)} = D^{(+,-)} \Phi^{(s_1, s_2)} = 0$ restrict effectively the superfield to depend on $USp(4)/(SU(2) \times U(1))$ harmonic variables rather than the ones on the $USp(4)/(U(1) \times U(1))$ coset. Therefore these constraints can be effectively resolved by considering the $USp(4)/(SU(2) \times U(1))$ harmonics. Nevertheless, in the present work we do not follow this way and work only with the $USp(4)/(U(1) \times U(1))$ harmonic variables introduced above.

4.1 $\mathcal{N}=4$ massive vector multiplet

Now we consider particular examples of superfields $\Phi^{(s_1, s_2)}$ with the lowest values of $U(1)$ charges satisfying (4.18). It is easy to see that the case $s_1 = s_2 = 0$ is trivial since such chargeless superfield is just a constant, $\Phi^{(0,0)} = \text{const}$. Therefore the first physically interesting example appears when $s_1 = s_2 = 1$,

$$\Phi^{(1,1)} \equiv W^{(+,+)} \tag{4.19}$$

One can check that for this superfield the equations in the last three lines in (4.18) follow from the other ones, while the relations in the first line are satisfied automatically. As a result, the superfield (4.19) obeys

$$D_\alpha^{(+,0)} W^{(+,+)} = D_\alpha^{(0,+)} W^{(+,+)} = \bar{D}_{\dot{\alpha}}^{(+,0)} W^{(+,+)} = \bar{D}_{\dot{\alpha}}^{(0,+)} W^{(+,+)} = 0, \tag{4.20}$$

$$D^{(++,0)} W^{(+,+)} = D^{(0,++)} W^{(+,+)} = D^{(+,+)} W^{(+,+)} = 0, \tag{4.21}$$

$$D^{(-,+)} W^{(+,+)} = D^{(+,-)} W^{(+,+)} = 0. \tag{4.22}$$

Equations (4.20) mean that the superfield $W^{(+,+)}$ is analytic. To resolve these constraints we first make the change of coordinates

$$x_A^m = x^m - i\theta^{(-,0)} \sigma^m \bar{\theta}^{(+,0)} - i\theta^{(+,0)} \sigma^m \bar{\theta}^{(-,0)} - i\theta^{(0,-)} \sigma^m \bar{\theta}^{(0,+)} - i\theta^{(0,+)} \sigma^m \bar{\theta}^{(0,-)}, \tag{4.23}$$

and then pass from τ to λ frame,⁸

$$\begin{aligned} D_\alpha^I &\rightarrow \mathcal{D}_\alpha^I = e^{\mathcal{Z}} D_\alpha^I e^{-\mathcal{Z}} = D_\alpha^I - (D_\alpha^I \mathcal{Z}), \\ \bar{D}_{\dot{\alpha}}^I &\rightarrow \bar{\mathcal{D}}_{\dot{\alpha}}^I = e^{\mathcal{Z}} \bar{D}_{\dot{\alpha}}^I e^{-\mathcal{Z}} = \bar{D}_{\dot{\alpha}}^I - (\bar{D}_{\dot{\alpha}}^I \mathcal{Z}), \\ W^{(+,+)} &\rightarrow \mathcal{W}^{(+,+)} = e^{\mathcal{Z}} W^{(+,+)}, \end{aligned} \quad (4.24)$$

where

$$\mathcal{Z} = iz\theta^{(+,0)\alpha}\theta_{\alpha}^{(-,0)} + iz\theta^{(0,+) \alpha}\theta_{\alpha}^{(0,-)} + i\bar{z}\bar{\theta}_{\dot{\alpha}}^{(+,0)}\bar{\theta}^{(-,0)\dot{\alpha}} + i\bar{z}\bar{\theta}_{\dot{\alpha}}^{(0,+)}\bar{\theta}^{(0,-)\dot{\alpha}} \quad (4.25)$$

is a bridge superfield. In the analytic coordinates (4.23) the derivatives (4.2) read

$$\begin{aligned} \mathcal{D}_\alpha^{(+,0)} &= \frac{\partial}{\partial\theta^{(-,0)\alpha}}, & \mathcal{D}_\alpha^{(-,0)} &= -\frac{\partial}{\partial\theta^{(0,+) \alpha}} + 2i(\sigma^m \bar{\theta}^{(-,0)})_\alpha \partial_m + 2iz\theta_\alpha^{(-,0)}, \\ \mathcal{D}_\alpha^{(0,+)} &= \frac{\partial}{\partial\theta^{(0,-)\alpha}}, & \mathcal{D}_\alpha^{(0,-)} &= -\frac{\partial}{\partial\theta^{(0,+) \alpha}} + 2i(\sigma^m \bar{\theta}^{(0,-)})_\alpha \partial_m + 2iz\theta_\alpha^{(0,-)}, \\ \bar{\mathcal{D}}_{\dot{\alpha}}^{(+,0)} &= \frac{\partial}{\partial\bar{\theta}^{(-,0)\dot{\alpha}}}, & \bar{\mathcal{D}}_{\dot{\alpha}}^{(-,0)} &= -\frac{\partial}{\partial\bar{\theta}^{(+,0)\dot{\alpha}}} - 2i(\theta^{(-,0)} \sigma^m)_{\dot{\alpha}} \partial_m - 2i\bar{z}\bar{\theta}_{\dot{\alpha}}^{(-,0)}, \\ \bar{\mathcal{D}}_{\dot{\alpha}}^{(0,+)} &= \frac{\partial}{\partial\bar{\theta}^{(0,-)\dot{\alpha}}}, & \bar{\mathcal{D}}_{\dot{\alpha}}^{(0,-)} &= -\frac{\partial}{\partial\bar{\theta}^{(0,+)\dot{\alpha}}} - 2i(\theta^{(0,-)} \sigma^m)_{\dot{\alpha}} \partial_m - 2i\bar{z}\bar{\theta}_{\dot{\alpha}}^{(0,-)}. \end{aligned} \quad (4.26)$$

Since the derivatives $\mathcal{D}_\alpha^{(+,0)}$, $\mathcal{D}_\alpha^{(0,+)}$, $\bar{\mathcal{D}}_{\dot{\alpha}}^{(+,0)}$, $\bar{\mathcal{D}}_{\dot{\alpha}}^{(0,+)}$ are short, the superfield $\mathcal{W}^{(+,+)}$ depends only on the analytic coordinates,

$$\mathcal{W}^{(+,+)} = \mathcal{W}^{(+,+)}(x_A^m, \theta_\alpha^{(+,0)}, \theta_\alpha^{(0,+)}, \bar{\theta}_{\dot{\alpha}}^{(+,0)}, \bar{\theta}_{\dot{\alpha}}^{(0,+)}, u). \quad (4.27)$$

To solve the constraints (4.21,4.22) we rewrite the covariant harmonic derivatives in the analytic coordinates in λ frame (omitting the terms vanishing on the analytic superfields),

$$\begin{aligned} \mathcal{D}^{(++ ,0)} &= D^{(++ ,0)} - 2i(\theta^{(+,0)} \sigma^m \bar{\theta}^{(+,0)}) \frac{\partial}{\partial x_A^m} - iz(\theta^{(+,0)})^2 - i\bar{z}(\bar{\theta}^{(+,0)})^2, \\ \mathcal{D}^{(0,++)} &= D^{(0,++)} - 2i(\theta^{(0,+) } \sigma^m \bar{\theta}^{(0,+)}) \frac{\partial}{\partial x_A^m} - iz(\theta^{(0,+)})^2 - i\bar{z}(\bar{\theta}^{(0,+)})^2, \\ \mathcal{D}^{(+,+)} &= D^{(+,+)} - 2i(\theta^{(+,0)} \sigma^m \bar{\theta}^{(0,+) } + \theta^{(0,+) } \sigma^m \bar{\theta}^{(+,0)}) \frac{\partial}{\partial x_A^m} \\ &\quad - 2iz(\theta^{(+,0)} \theta^{(0,+)}) - 2i\bar{z}(\bar{\theta}^{(+,0)} \bar{\theta}^{(0,+)}) , \\ \mathcal{D}^{(-,+)} &= D^{(-,+)} - \theta_\alpha^{(0,+) } \frac{\partial}{\partial\theta_\alpha^{(+,0)}} - \bar{\theta}_{\dot{\alpha}}^{(0,+)} \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^{(+,0)}}, \\ \mathcal{D}^{(+,-)} &= D^{(+,-)} + \theta_\alpha^{(+,0)} \frac{\partial}{\partial\theta_\alpha^{(0,+)}} + \bar{\theta}_{\dot{\alpha}}^{(+,0)} \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^{(0,+)}}. \end{aligned} \quad (4.28)$$

⁸Here we use the terminology of [6] used for similar constructions in $\mathcal{N}=2$ harmonic superspace. In fact, such passing from τ to λ frame is nothing but the use of two different representations for Grassmann and harmonic derivatives which are related by the unitary operator $e^{\mathcal{Z}}$.

The full component structure of the superfield (4.27) is very long. However, on the equations of motion (4.21,4.22) all auxiliary fields vanish and the component decomposition appears pretty short,

$$\begin{aligned}
\mathcal{W}^{(+,+)} = & u_{[i}^{(+,0)} u_{j]}^{(0,+)} f^{ij} + \theta^{(+,0)\alpha} \psi_{\alpha}^i u_i^{(0,+)} - \theta^{(0,+) \alpha} \psi_{\alpha}^i u_i^{(+,0)} \\
& + \bar{\theta}_{\dot{\alpha}}^{(+,0)} \bar{\chi}^{i\dot{\alpha}} u_i^{(0,+)} - \bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{\chi}^{i\dot{\alpha}} u_i^{(+,0)} \\
& + iz(\theta^{(+,0)})^2 f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,+)} + iz(\theta^{(0,+)})^2 f^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,-)} \\
& + i\bar{z}(\bar{\theta}^{(+,0)})^2 f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,+)} + i\bar{z}(\bar{\theta}^{(0,+)})^2 f^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,-)} \\
& + (iz\theta^{(+,0)}\theta^{(0,+)} + i\bar{z}\bar{\theta}^{(+,0)}\bar{\theta}^{(0,+)}) f^{ij} (u_{[i}^{(+,0)} u_{j]}^{(-,0)} - u_{[i}^{(0,+)} u_{j]}^{(0,-)}) \\
& + 2i\theta^{(+,0)}\sigma^m\bar{\theta}^{(0,+)}\partial_m f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,+)} + 2i\theta^{(0,+)}\sigma^m\bar{\theta}^{(0,+)}\partial_m f^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,-)} \\
& + i(\theta_{\alpha}^{(+,0)}\bar{\theta}_{\dot{\alpha}}^{(0,+)} + \theta_{\alpha}^{(0,+)}\bar{\theta}_{\dot{\alpha}}^{(+,0)})\sigma^{m\alpha\dot{\alpha}}\partial_m f^{ij} (u_{[i}^{(+,0)} u_{j]}^{(-,0)} - u_{[i}^{(0,+)} u_{j]}^{(0,-)}) \\
& + \theta_{\alpha}^{(+,0)}\theta_{\beta}^{(0,+)} F^{\alpha\beta} + \bar{\theta}_{\dot{\alpha}}^{(+,0)}\bar{\theta}_{\dot{\beta}}^{(0,+)} \bar{G}^{\dot{\alpha}\dot{\beta}} + \theta_{\alpha}^{(+,0)}\bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{A}^{\alpha\dot{\alpha}} + \theta_{\alpha}^{(0,+)}\bar{\theta}_{\dot{\alpha}}^{(+,0)} A^{\alpha\dot{\alpha}} \\
& - iz(\theta^{(+,0)})^2 \theta^{(0,+)\alpha} \psi_{\alpha}^i u_i^{(-,0)} - iz(\theta^{(+,0)})^2 \bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{\chi}^{i\dot{\alpha}} u_i^{(-,0)} \\
& + iz(\theta^{(0,+)})^2 \theta^{(+,0)\alpha} \psi_{\alpha}^i u_i^{(0,-)} + iz(\theta^{(0,+)})^2 \bar{\theta}_{\dot{\alpha}}^{(+,0)} \bar{\chi}^{i\dot{\alpha}} u_i^{(0,-)} \\
& - i\bar{z}(\bar{\theta}^{(+,0)})^2 \theta^{(0,+)\alpha} \psi_{\alpha}^i u_i^{(-,0)} - i\bar{z}(\bar{\theta}^{(+,0)})^2 \bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{\chi}^{i\dot{\alpha}} u_i^{(0,-)} \\
& + i\bar{z}(\bar{\theta}^{(0,+)})^2 \theta^{(+,0)\alpha} \psi_{\alpha}^i u_i^{(0,-)} + i\bar{z}(\bar{\theta}^{(0,+)})^2 \bar{\theta}_{\dot{\alpha}}^{(+,0)} \bar{\chi}^{i\dot{\alpha}} u_i^{(0,-)} \\
& + 2i\theta^{(+,0)\alpha}\theta^{(0,+) \beta}\bar{\theta}^{(+,0)\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^m\partial_m\psi_{\beta}^i u_i^{(-,0)} + 2i\theta^{(+,0)\beta}\theta^{(0,+) \alpha}\bar{\theta}^{(0,+) \dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^m\partial_m\psi_{\beta}^i u_i^{(0,-)} \\
& + 2i\bar{\theta}^{(+,0)\dot{\alpha}}\bar{\theta}^{(0,+) \dot{\beta}}\theta^{(+,0)\alpha}\sigma_{\alpha\dot{\alpha}}^m\partial_m\bar{\chi}_{\dot{\beta}}^i u_i^{(-,0)} + 2i\bar{\theta}^{(+,0)\dot{\beta}}\bar{\theta}^{(0,+) \dot{\alpha}}\theta^{(0,+) \alpha}\sigma_{\alpha\dot{\alpha}}^m\partial_m\bar{\chi}_{\dot{\beta}}^i u_i^{(0,-)} \\
& - z^2(\theta^{(+,0)})^2(\theta^{(0,+)})^2 f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,-)} - \bar{z}^2(\bar{\theta}^{(+,0)})^2(\bar{\theta}^{(0,+)})^2 f^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,-)} \\
& - z\bar{z}[(\theta^{(+,0)})^2(\bar{\theta}^{(0,+)})^2 + (\theta^{(0,+)})^2(\bar{\theta}^{(+,0)})^2] f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,-)} \\
& - 2z[(\theta^{(+,0)})^2\theta_{\alpha}^{(0,+)}\bar{\theta}_{\dot{\alpha}}^{(0,+)} + (\theta^{(0,+)})^2\theta_{\alpha}^{(+,0)}\bar{\theta}_{\dot{\alpha}}^{(+,0)}]\sigma^{m\alpha\dot{\alpha}}\partial_m f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,-)} \\
& - 2\bar{z}[(\bar{\theta}^{(+,0)})^2\theta_{\alpha}^{(0,+)}\bar{\theta}_{\dot{\alpha}}^{(0,+)} + (\bar{\theta}^{(0,+)})^2\theta_{\alpha}^{(+,0)}\bar{\theta}_{\dot{\alpha}}^{(+,0)}]\sigma^{m\alpha\dot{\alpha}}\partial_m f^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,-)} \\
& + 4\theta_{\alpha}^{(+,0)}\theta_{\beta}^{(0,+)}\bar{\theta}_{\dot{\alpha}}^{(+,0)}\bar{\theta}_{\dot{\beta}}^{(0,+)}\sigma^{m\alpha\dot{\alpha}}\sigma^{n\beta\dot{\beta}}\partial_m\partial_n f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,-)}. \tag{4.29}
\end{aligned}$$

The scalar fields in (4.29) satisfy Klein-Gordon equation,

$$(\square + z\bar{z})f^{ij} = 0, \tag{4.30}$$

while for the spinors we have Dirac equations with the correct signs in the mass terms owing to (3.8),

$$i\sigma_{\alpha\dot{\alpha}}^m\partial_m\psi^{i\alpha} - iz\bar{\chi}^{i\dot{\alpha}} = 0, \quad i\sigma_{\alpha\dot{\alpha}}^m\partial_m\bar{\chi}^{i\dot{\beta}} - i\bar{z}\psi_{\alpha}^i = 0. \tag{4.31}$$

The fields $F_{\alpha\beta}$, $A_{\alpha\dot{\alpha}}$ and $\bar{G}_{\dot{\alpha}\dot{\beta}}$, $\bar{A}_{\alpha\dot{\alpha}}$ are related to each other as

$$\begin{aligned}
i\sigma^{m\alpha}_{\dot{\alpha}}\partial_m F_{\alpha\beta} + izA_{\beta\dot{\alpha}} &= 0, & i\sigma^{m\alpha}_{\dot{\alpha}}\partial_m A_{\beta\dot{\alpha}} + i\bar{z}F_{\alpha\beta} &= 0, \\
i\sigma^{m\alpha}_{\dot{\alpha}}\partial_m \bar{G}_{\dot{\alpha}\dot{\beta}} + i\bar{z}\bar{A}_{\alpha\dot{\beta}} &= 0, & i\sigma^{m\alpha}_{\dot{\alpha}}\partial_m \bar{A}_{\alpha\dot{\beta}} + iz\bar{G}_{\dot{\alpha}\dot{\beta}} &= 0, \\
\sigma_{\alpha\dot{\alpha}}^m\partial_m A^{\alpha\dot{\alpha}} &= 0, & \sigma_{\alpha\dot{\alpha}}^m\partial_m \bar{A}^{\alpha\dot{\alpha}} &= 0.
\end{aligned} \tag{4.32}$$

Equations (4.32) mean that the fields $A_{\alpha\dot{\alpha}}$ and $\bar{A}_{\alpha\dot{\alpha}}$ obey the relations

$$\begin{aligned} (\square + z\bar{z})A_{\alpha\dot{\alpha}} &= 0, & \sigma_{\alpha\dot{\alpha}}^m \partial_m A^{\alpha\dot{\alpha}} &= 0, \\ (\square + z\bar{z})\bar{A}_{\alpha\dot{\alpha}} &= 0, & \sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{A}^{\alpha\dot{\alpha}} &= 0 \end{aligned} \quad (4.33)$$

and, hence, describe complex massive vector field. We conclude that (4.29) represents a superfield realization of $\mathcal{N}=4$ massive vector multiplet,

$$\begin{aligned} (\square + m^2)f^{ij} &= 0 \quad (\Omega_{ij}f^{ij} = 0) & 5 \text{ complex scalars,} \\ i\sigma_{\alpha\dot{\alpha}}^m \partial_m \psi^{i\alpha} + m\bar{\chi}^{i\dot{\alpha}} &= 0 \\ i\sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{\chi}^{i\dot{\beta}} - m\psi_{\alpha}^i &= 0 & 4 \text{ Dirac spinors,} \\ (\square + m^2)A_{\alpha\dot{\alpha}} &= 0, \quad \sigma_{\alpha\dot{\alpha}}^m \partial_m A_{\alpha\dot{\alpha}} = 0 \\ (\square + m^2)\bar{A}_{\alpha\dot{\alpha}} &= 0, \quad \sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{A}_{\alpha\dot{\alpha}} = 0 & 1 \text{ massive complex vector.} \end{aligned} \quad (4.34)$$

Let us now consider in short the second type of separation of superparticle constraints (4.9) on the examples of a superfield $W^{(+,-)}$. The Grassmann derivatives in (4.9) should annihilate this superfield,

$$D_{\alpha}^{(+,0)}W^{(+,-)} = D_{\alpha}^{(0,-)}W^{(+,-)} = \bar{D}_{\dot{\alpha}}^{(+,0)}W^{(+,-)} = \bar{D}_{\dot{\alpha}}^{(0,-)}W^{(+,-)} = 0. \quad (4.35)$$

We impose also the following harmonic constraints

$$D^{(++,0)}W^{(+,-)} = D^{(0,--)}W^{(+,-)} = D^{(+,-)}W^{(+,-)} = D^{(-,-)}W^{(+,-)} = D^{(+,+)}W^{(+,-)} = 0, \quad (4.36)$$

since harmonic derivatives in (4.36) leave the set of the constraints (4.35) invariant.

To solve the constraints (4.35) we consider this superfield in analytic coordinates,

$$x_A'^m = x^m - i\theta^{(-,0)}\sigma^m\bar{\theta}^{(+,0)} - i\theta^{(+,0)}\sigma^m\bar{\theta}^{(-,0)} + i\theta^{(0,-)}\sigma^m\bar{\theta}^{(0,+)} + i\theta^{(0,+)}\sigma^m\bar{\theta}^{(0,-)} \quad (4.37)$$

and pass from τ to λ frame by the rules (4.24) with the bridge

$$\mathcal{Z} = iz\theta^{(+,0)\alpha}\theta_{\alpha}^{(-,0)} - iz\theta^{(0,+)\alpha}\theta_{\alpha}^{(0,-)} + i\bar{z}\bar{\theta}_{\dot{\alpha}}^{(+,0)}\bar{\theta}^{(-,0)\dot{\alpha}} - i\bar{z}\bar{\theta}_{\dot{\alpha}}^{(0,+)}\bar{\theta}^{(0,-)\dot{\alpha}}. \quad (4.38)$$

The component structure of the superfield $\mathcal{W}^{(+,-)}$ is found in a similar way,

$$\begin{aligned} \mathcal{W}^{(+,-)} &= u_{[i}^{(+,0)}u_{j]}^{(0,-)}f^{ij} + \theta^{(+,0)\alpha}\psi_{\alpha}^i u_i^{(0,-)} - \theta^{(0,-)\alpha}\psi_{\alpha}^i u_i^{(+,0)} + \bar{\theta}_{\dot{\alpha}}^{(+,0)}\bar{\chi}^{i\dot{\alpha}}u_i^{(0,-)} - \bar{\theta}_{\dot{\alpha}}^{(0,-)}\bar{\chi}^{i\dot{\alpha}}u_i^{(+,0)} \\ &+ iz(\theta^{(+,0)})^2 f^{ij}u_{[i}^{(-,0)}u_{j]}^{(0,-)} - iz(\theta^{(0,-)})^2 f^{ij}u_{[i}^{(+,0)}u_{j]}^{(0,+)} \\ &+ i\bar{z}(\bar{\theta}^{(+,0)})^2 f^{ij}u_{[i}^{(-,0)}u_{j]}^{(0,-)} - i\bar{z}(\bar{\theta}^{(0,-)})^2 f^{ij}u_{[i}^{(+,0)}u_{j]}^{(0,+)} \\ &+ (iz\theta^{(+,0)}\theta^{(0,-)} + i\bar{z}\bar{\theta}^{(+,0)}\bar{\theta}^{(0,-)})f^{ij}(u_{[i}^{(+,0)}u_{j]}^{(-,0)} - u_{[i}^{(0,+)}u_{j]}^{(0,-)}) \\ &+ 2i\theta^{(+,0)}\sigma^m\bar{\theta}^{(+,0)}\partial_m f^{ij}u_{[i}^{(-,0)}u_{j]}^{(0,-)} - 2i\theta^{(0,-)}\sigma^m\bar{\theta}^{(0,-)}\partial_m f^{ij}u_{[i}^{(+,0)}u_{j]}^{(0,+)} \\ &+ i(\theta_{\alpha}^{(+,0)}\bar{\theta}_{\dot{\alpha}}^{(0,-)} + \theta_{\alpha}^{(0,-)}\bar{\theta}_{\dot{\alpha}}^{(+,0)})\sigma^{m\alpha\dot{\alpha}}\partial_m f^{ij}(u_{[i}^{(+,0)}u_{j]}^{(-,0)} - u_{[i}^{(0,+)}u_{j]}^{(0,-)}) \\ &+ \theta_{\alpha}^{(+,0)}\theta_{\beta}^{(0,-)}F^{\alpha\beta} + \bar{\theta}_{\dot{\alpha}}^{(+,0)}\bar{\theta}_{\dot{\beta}}^{(0,-)}\bar{G}^{\dot{\alpha}\dot{\beta}} + \theta_{\alpha}^{(+,0)}\bar{\theta}_{\dot{\alpha}}^{(0,-)}\bar{A}^{\alpha\dot{\alpha}} + \theta_{\alpha}^{(0,-)}\bar{\theta}_{\dot{\alpha}}^{(+,0)}A^{\alpha\dot{\alpha}} \\ &+ \dots, \end{aligned} \quad (4.39)$$

where dots stand for the terms with θ 's to the third and fourth powers. All components here depend on x_A^m and satisfy the free equations of motion (4.34).

Analogously, one can consider the superfield $W^{(-,+)}$ constrained by

$$\begin{aligned} D_\alpha^{(-,0)}W^{(+,-)} &= D_\alpha^{(0,+)}W^{(+,-)}\bar{D}_{\dot{\alpha}}^{(-,0)}W^{(+,-)} = \bar{D}_{\dot{\alpha}}^{(0,+)}W^{(+,-)} = 0, \\ D^{(--,0)}W^{(+,-)} &= D^{(0,++)}W^{(+,-)} = D^{(-,+)}W^{(+,-)} = D^{(-,-)}W^{(+,-)} = D^{(+,+)}W^{(+,-)} = 0. \end{aligned} \quad (4.40)$$

The component structure of $W^{(-,+)}$ is similar to (4.39), but the values of the $U(1)$ charges are swapped.

5 Gupta-Bleuler quantization of massless superparticle

We turn to the massless $\mathcal{N}=4$ superparticle by considering the model (3.1) in the massless limit, i.e.

$$m = z = \bar{z} = 0. \quad (5.1)$$

The Lagrangian (3.1) reduces to

$$L_1 = -\frac{1}{2}e^{-1}\dot{\omega}^m\dot{\omega}_m. \quad (5.2)$$

We point out that the massless superparticle has $U(4)$ R-symmetry group rather than $USp(4)$. Therefore, it is naturally to extend this model with the $SU(4)$ harmonics. This case was already studied in [18], where the superfield description of $\mathcal{N}=4$ vector multiplet was given. Therefore in the present work we study the $\mathcal{N}=4$ massless superparticle extended by $USp(4)$ harmonic variables. Exactly this feature allows us to get some new insight on the problem of superfield formulation of $\mathcal{N}=4$ SYM model in harmonic superspace.

The Lagrangian of $\mathcal{N}=4$ harmonic superparticle reads

$$L_{\mathcal{N}=4} = L_1 + L_{USp(4)}, \quad (5.3)$$

where L_1 and $L_{USp(4)}$ are defined in (5.2) and (3.19), respectively. The further quantization of this model is straightforward and can be read from the above considerations in the limit (5.1). Here we mention only the new features.

In the massless case the algebra of covariant spinor derivatives (4.2) has non-trivial anticommutation relations given only in the last line in (4.3). Therefore, in the Gupta-Bleuler quantization approach, there are eight ways of separation of corresponding con-

straints,

$$\{D_{\alpha}^{(0,+)}, D_{\alpha}^{(0,-)}, \bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(-,0)}\} \cup \{D_{\alpha}^{(+,0)}, D_{\alpha}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\}, \quad (5.4a)$$

$$\{D_{\alpha}^{(+,0)}, D_{\alpha}^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}\} \cup \{D_{\alpha}^{(-,0)}, D_{\alpha}^{(0,-)}, \bar{D}_{\dot{\alpha}}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\}, \quad (5.4b)$$

$$\{D_{\alpha}^{(+,0)}, D_{\alpha}^{(0,-)}, \bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\} \cup \{D_{\alpha}^{(-,0)}, D_{\alpha}^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}\}, \quad (5.4c)$$

$$\{D_{\alpha}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}, D_{\alpha}^{(-,0)}, D_{\alpha}^{(0,-)}\} \cup \{\bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(-,0)}, D_{\alpha}^{(0,+)}\}, \quad (5.4d)$$

$$\{D_{\alpha}^{(+,0)}, D_{\alpha}^{(0,+)}, D_{\alpha}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}\} \cup \{\bar{D}_{\dot{\alpha}}^{(+,0)}, D_{\alpha}^{(0,-)}, \bar{D}_{\dot{\alpha}}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\}, \quad (5.4e)$$

$$\{D_{\alpha}^{(+,0)}, D_{\alpha}^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(+,0)}, D_{\alpha}^{(0,-)}\} \cup \{D_{\alpha}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\}, \quad (5.4f)$$

$$\{\bar{D}_{\dot{\alpha}}^{(-,0)}, D_{\alpha}^{(0,+)}, D_{\alpha}^{(-,0)}, D_{\alpha}^{(0,-)}\} \cup \{\bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}, D_{\alpha}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\}, \quad (5.4g)$$

$$\{D_{\alpha}^{(+,0)}, D_{\alpha}^{(0,+)}, D_{\alpha}^{(-,0)}, D_{\alpha}^{(0,-)}\} \cup \{\bar{D}_{\dot{\alpha}}^{(+,0)}, \bar{D}_{\dot{\alpha}}^{(0,+)}, \bar{D}_{\dot{\alpha}}^{(-,0)}, \bar{D}_{\dot{\alpha}}^{(0,-)}\}. \quad (5.4h)$$

As before, we assume that the state is described by a superfield $\Phi^{(s_1, s_2)}$ subject to (4.5). To take into account the constraints (5.4) we claim

$$\{D_{\alpha, \dot{\alpha}}\} \Phi^{(s_1, s_2)} = 0, \quad (5.5)$$

where $\{D_{\alpha, \dot{\alpha}}\}$ are covariant spinor derivatives from one of the subsets in (5.4). We have to impose also the harmonic constraints,

$$\{D^A\} \Phi^{(s_1, s_2)} = 0, \quad (5.6)$$

where $\{D^A\}$ is a subset of harmonic derivatives (2.16) which leave the set of Grassmann derivatives $\{D_{\alpha, \dot{\alpha}}\}$ invariant. Apart from these constraints there are also κ -symmetry ones with the generators (4.13). In the massless case they lead to the following equations

$$\sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{D}^{I\dot{\alpha}} \Phi^{(s_1, s_2)} = 0, \quad \sigma_{\alpha\dot{\alpha}}^m \partial_m D^{I\alpha} \Phi^{(s_1, s_2)} = 0. \quad (5.7)$$

Using the algebra of covariant spinor derivatives one can show that the constraints (5.7) follow from

$$\bar{D}_{\dot{\alpha}}^I \bar{D}^{J\dot{\alpha}} \Phi^{(s_1, s_2)} = 0, \quad D^{I\alpha} D_{\alpha}^J \Phi^{(s_1, s_2)} = 0, \quad (5.8)$$

where the indices I, J have the values (3.24). As explained in the Appendix 2, the constraints (5.8) originate from the bosonic version of κ -symmetry. We point out that not all the constraints (5.6, 5.8) are independent, however they are all first-class. Some of them become trivial when definite subset in (5.4) is chosen.

As a result, all the first-class and one half of second-class constraints are taken into account by the equations (5.5)–(5.8) for the superfield $\Phi^{(s_1, s_2)}$. In the following subsections we consider the particular examples of massless representations of $\mathcal{N}=4$ superalgebra on such superfields with lowest values of $U(1)$ charges s_1, s_2 , which correspond to different ways of separations of constraints (5.4) and describe different multiplets.

5.1 Supergauge multiplet

Regarding to the separations of constraints (5.4a)–(5.4c), there are two essentially different superfield realizations of $\mathcal{N}=4$ supergauge multiplets. In the next subsections we consider both of them separately.

5.1.1 Chargeless superfield representation

According to the separation of constraints (5.4a), the superfield $\Phi^{(s_1, s_2)}$ should be annihilated by the following Grassmann derivatives

$$D_\alpha^{(0,+)}\Phi^{(s_1, s_2)} = D_\alpha^{(0,-)}\Phi^{(s_1, s_2)} = \bar{D}_{\dot{\alpha}}^{(+,0)}\Phi^{(s_1, s_2)} = \bar{D}_{\dot{\alpha}}^{(-,0)}\Phi^{(s_1, s_2)} = 0. \quad (5.9)$$

As follows from (5.8), the linearity conditions read

$$\begin{aligned} (D^{(+,0)})^2\Phi^{(s_1, s_2)} &= (D^{(-,0)})^2\Phi^{(s_1, s_2)} = (D^{(+,0)}D^{(-,0)})\Phi^{(s_1, s_2)} = 0, \\ (\bar{D}^{(0,+)})^2\Phi^{(s_1, s_2)} &= (\bar{D}^{(0,-)})^2\Phi^{(s_1, s_2)} = (\bar{D}^{(0,+)}\bar{D}^{(0,-)})\Phi^{(s_1, s_2)} = 0. \end{aligned} \quad (5.10)$$

We have to impose also the harmonic constraints, to fix the dynamics over the harmonic variables. Only the operators $D^{(++,0)}$, $D^{(--,0)}$, $D^{(0,++)}$, $D^{(0,--)}$ leave the set of Grassmann derivatives in (5.9) invariant. Therefore we claim

$$D^{(++,0)}\Phi^{(s_1, s_2)} = D^{(--,0)}\Phi^{(s_1, s_2)} = D^{(0,++)}\Phi^{(s_1, s_2)} = D^{(0,--)}\Phi^{(s_1, s_2)} = 0. \quad (5.11)$$

According to the algebra (A.3), $[D^{(++,0)}, D^{(--,0)}] = S_1$, $[D^{(0,++)}, D^{(0,--)}] = S_2$, the constraints (5.11) imply $s_1 = s_2 = 0$. Hence, the state under considerations is realized by chargeless superfield $\Phi^{(0,0)} \equiv W_1$.

However, the constraints (5.11) do not fix the harmonic dynamics completely. Therefore we impose also the quadratic harmonic constraint,

$$D^{(+,+)}D^{(+,+)}W_1 = 0. \quad (5.12)$$

As a consequence of the algebra (A.3), the operators $D^{(-,-)}D^{(+,+)}$, $D^{(+,-)}D^{(-,+)}$ annihilate the state as well and the whole operator (4.7) vanishes on this superfield,

$$X W_1 = 0. \quad (5.13)$$

Therefore we are convinced that the constraints (5.11, 5.12) are sufficient to fix the dynamics over the harmonic variables and to eliminate all unphysical degrees of freedom in this superfield.

Let us rewrite all these constraints for the superfield W_1 in a single list,

$$\begin{aligned} D_\alpha^{(0,+)}W_1 &= D_\alpha^{(0,-)}W_1 = \bar{D}_{\dot{\alpha}}^{(+,0)}W_1 = \bar{D}_{\dot{\alpha}}^{(-,0)}W_1 = 0, \\ D^{(++,0)}W_1 &= D^{(--,0)}W_1 = D^{(0,++)}W_1 = D^{(0,--)}W_1 = 0, \\ D^{(+,+)}D^{(+,+)}W_1 &= 0. \end{aligned} \quad (5.14)$$

The linearity constraints (5.10) are not in this list since they are not independent but follow from (5.14).

To solve the constraints in the first line of (5.14) we pass to the analytic coordinates,

$$x_A^m = x^m - i\theta^{(0,-)}\sigma^m\bar{\theta}^{(0,+)} + i\theta^{(0,+)}\sigma^m\bar{\theta}^{(0,-)} - i\theta^{(+,0)}\sigma^m\bar{\theta}^{(-,0)} + i\theta^{(-,0)}\sigma^m\bar{\theta}^{(+,0)}, \quad (5.15)$$

in which the Grassmann derivatives in (5.9) become short,

$$D_\alpha^{(0,\pm)} = \pm \frac{\partial}{\partial \theta^{(0,\mp)}_\alpha}, \quad \bar{D}_{\dot{\alpha}}^{(\pm,0)} = \pm \frac{\partial}{\partial \bar{\theta}^{(\mp,0)}_{\dot{\alpha}}}. \quad (5.16)$$

Therefore the superfield W_1 depends only on one half of Grassmann variables,

$$W_1 = W_1(x_A^m, \theta_\alpha^{(+,0)}, \theta_\alpha^{(-,0)}, \bar{\theta}_{\dot{\alpha}}^{(0,+)}, \bar{\theta}_{\dot{\alpha}}^{(0,-)}, u). \quad (5.17)$$

Let us rewrite also the harmonic derivatives in (5.14) in such analytic coordinates (we omit the terms acting trivially on W_1),

$$D_A^{(\pm\pm,0)} = D^{(\pm\pm,0)} + \theta^{(\pm,0)} \frac{\partial}{\partial \theta^{(\mp,0)}}, \quad D_A^{(0,\pm\pm)} = D^{(0,\pm\pm)} + \theta^{(0,\pm)} \frac{\partial}{\partial \theta^{(0,\mp)}}, \quad (5.18)$$

$$\begin{aligned} D_A^{(+,+)} &= D^{(+,+)} - 2i(\theta^{(+,0)} \sigma^m \bar{\theta}^{(0,+)} - \theta^{(0,+)} \sigma^m \bar{\theta}^{(+,0)}) \partial_m \\ &\quad + \theta^{(0,+)} \frac{\partial}{\partial \theta^{(-,0)}} + \bar{\theta}^{(+,0)} \frac{\partial}{\partial \bar{\theta}^{(0,-)}}. \end{aligned} \quad (5.19)$$

Since the operators (5.18) do not contain the spatial derivatives, the constraints in the second line of (5.14) are not kinematical but purely algebraical for the components of W_1 . In fact, these constraints show that W_1 depends effectively on $USp(4)/(SU(2) \times SU(2))$ harmonic variables since the derivatives in (5.11) form two $su(2)$ algebras. In principle, such harmonic constraints (5.11) can be solved manifestly by choosing the appropriate harmonic variables. As a result, only the constraint in the last line of (5.14) is true equation of motion for the superfield W_1 .

Finally, we impose also the reality constraint,

$$\widetilde{W}_1 = W_1, \quad (5.20)$$

where the conjugation is defined in (2.20).

Now we give the solution of all the constraints for the superfield W_1 ,

$$\begin{aligned} W_1 &= \phi + i f^{ij} (u_{[i}^{(+,0)} u_{j]}^{(-,0)} - u_{[i}^{(0,+)} u_{j]}^{(0,-)}) \\ &\quad + i \theta^{(+,0)\alpha} \psi_\alpha^i u_i^{(-,0)} - i \theta^{(-,0)\alpha} \psi_\alpha^i u_i^{(+,0)} + i \bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{\psi}^{i\dot{\alpha}} u_i^{(0,-)} - i \bar{\theta}_{\dot{\alpha}}^{(0,-)} \bar{\psi}^{i\dot{\alpha}} u_i^{(0,+)} \\ &\quad + \theta_\alpha^{(+,0)} \theta_\beta^{(-,0)} F^{(\alpha\beta)} + \bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{\theta}_{\dot{\beta}}^{(0,-)} \bar{F}^{(\dot{\alpha}\dot{\beta})} \\ &\quad + 4\theta_\alpha^{(+,0)} \bar{\theta}_{\dot{\alpha}}^{(0,+)} \partial_{\alpha\dot{\alpha}} f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,-)} + 4\theta_\alpha^{(-,0)} \bar{\theta}_{\dot{\alpha}}^{(0,-)} \partial_{\alpha\dot{\alpha}} f^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,+)} \\ &\quad - 4\theta_\alpha^{(+,0)} \bar{\theta}_{\dot{\alpha}}^{(0,-)} \partial_{\alpha\dot{\alpha}} f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,+)} - 4\theta_\alpha^{(-,0)} \bar{\theta}_{\dot{\alpha}}^{(0,+)} \partial_{\alpha\dot{\alpha}} f^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,-)} \\ &\quad - 2\theta^{(+,0)\alpha} \theta^{(-,0)\beta} \bar{\theta}^{(0,+)\dot{\alpha}} \partial_{(\alpha\dot{\alpha}} \psi_{\beta)}^i u_i^{(0,-)} + 2\theta^{(+,0)\alpha} \theta^{(-,0)\beta} \bar{\theta}^{(0,-)\dot{\alpha}} \partial_{(\alpha\dot{\alpha}} \psi_{\beta)}^i u_i^{(0,+)} \\ &\quad - 2\theta^{(+,0)\alpha} \bar{\theta}^{(0,+)\dot{\beta}} \bar{\theta}^{(0,-)\dot{\alpha}} \partial_{\alpha(\dot{\alpha}} \bar{\psi}_{\dot{\beta})}^i u_i^{(-,0)} + 2\theta^{(-,0)\alpha} \bar{\theta}^{(0,+)\dot{\beta}} \bar{\theta}^{(0,-)\dot{\alpha}} \partial_{\alpha(\dot{\alpha}} \bar{\psi}_{\dot{\beta})}^i u_i^{(+,0)} \\ &\quad + 4\theta_\alpha^{(+,0)} \theta_\beta^{(-,0)} \bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{\theta}_{\dot{\beta}}^{(0,-)} \partial^{\alpha\dot{\alpha}} \partial^{\beta\dot{\beta}} [\phi - i f^{ij} (u_{[i}^{(+,0)} u_{j]}^{(-,0)} - u_{[i}^{(0,+)} u_{j]}^{(0,-)})]. \end{aligned} \quad (5.21)$$

All component fields here depend only on x_A^m and satisfy the corresponding free equations of motion,

$$\begin{aligned}
\Box\phi &= 0 & 1 \text{ real scalar,} \\
\Box f^{ij} &= 0, \quad (f^{ij}\Omega_{ij} = 0) & 5 \text{ real scalars,} \\
\sigma^{m\alpha}{}_{\dot{\alpha}}\partial_m\psi_{\alpha}^i &= 0, \quad \sigma^m{}_{\alpha}{}^{\dot{\alpha}}\partial_m\bar{\psi}_{\dot{\alpha}}^i = 0 & 4 \text{ Weyl spinors,} \\
\sigma^{m\alpha}{}_{\dot{\alpha}}\partial_m F_{(\alpha\beta)} &= 0, \quad \sigma^m{}_{\alpha}{}^{\dot{\alpha}}\partial_m \bar{F}_{(\dot{\alpha}\dot{\beta})} = 0 & 1 \text{ Maxwell field}
\end{aligned} \tag{5.22}$$

and reality conditions,

$$\bar{\phi} = \phi, \quad \bar{f}^{ij} = f^{ij}, \quad \bar{\psi}_{\alpha}^i = \bar{\psi}_{i\dot{\alpha}}, \quad \overline{F_{(\alpha\beta)}} = \bar{F}_{(\dot{\alpha}\dot{\beta})}. \tag{5.23}$$

As a result, the $\mathcal{N}=4$ SYM multiplet is embedded into the superfield W_1 .

Let us consider now the Grassmann derivatives in the second subset of (5.4a). We denote the superfield annihilated by these variables as W_2 . Similarly as W_1 , it satisfies the following constraints,

$$\begin{aligned}
\bar{D}_{\dot{\alpha}}^{(0,+)}W_2 &= \bar{D}_{\dot{\alpha}}^{(0,-)}W_2 = D_{\alpha}^{(+,0)}W_2 = D_{\alpha}^{(-,0)}W_2 = 0, \\
D^{(++,0)}W_2 &= D^{(--,0)}W_2 = D^{(0,++)}W_2 = D^{(0,--)}W_2 = 0, \\
D^{(-,-)}D^{(-,-)}W_2 &= 0, \\
\widetilde{W}_2 &= W_2.
\end{aligned} \tag{5.24}$$

The Grassmann constraints in (5.24) are solved by passing to the corresponding analytic coordinates,

$$W_2 = W_2(x_A^m, \bar{\theta}_{\dot{\alpha}}^{(+,0)}, \bar{\theta}_{\dot{\alpha}}^{(-,0)}, \theta_{\alpha}^{(0,+)}, \theta_{\alpha}^{(0,-)}, u), \tag{5.25}$$

where

$$x_A^m = x^m + i\theta^{(0,-)}\sigma^m\bar{\theta}^{(0,+)} - i\theta^{(0,+)}\sigma^m\bar{\theta}^{(0,-)} + i\theta^{(+,0)}\sigma^m\bar{\theta}^{(-,0)} - i\theta^{(-,0)}\sigma^m\bar{\theta}^{(+,0)}. \tag{5.26}$$

The component structure of W_2 is analogous to (5.21), but the values of $U(1)$ charges of superspace coordinates should be changed appropriately. Therefore, W_2 also describes the $\mathcal{N}=4$ SYM multiplet.

5.1.2 Charged superfield representation

Consider the separation of constraints (5.4b) on the example of a superfield $W^{(+,+)}$ with $U(1)$ charges $s_1 = s_2 = 1$. According to (5.4b) and (4.11), it satisfies similar equations as the massive vector multiplet considered in sect. 4.1,

$$D_{\alpha}^{(+,0)}W^{(+,+)} = D_{\alpha}^{(0,+)}W^{(+,+)} = \bar{D}_{\dot{\alpha}}^{(+,0)}W^{(+,+)} = \bar{D}_{\dot{\alpha}}^{(0,+)}W^{(+,+)} = 0, \tag{5.27}$$

$$D^{(++,0)}W^{(+,+)} = D^{(0,++)}W^{(+,+)} = D^{(+,+)}W^{(+,+)} = 0, \tag{5.28}$$

$$D^{(-,+)}W^{(+,+)} = D^{(+,-)}W^{(+,+)} = 0. \tag{5.29}$$

We point out that the harmonic constraints (5.29) are not kinematical, but they just show that $W^{(+,+)}$ depends effectively on $USp(4)/(SU(2) \times U(1))$ harmonic variables. These

constraints can be solved manifestly by the appropriate choice of harmonics, but we prefer to work here with the harmonic variables on the $USp(4)/(U(1) \times U(1))$ coset, introduced above.

There are also linearity conditions (5.8),

$$(D^{(-,0)})^2 W^{(+,+)} = (D^{(0,-)})^2 W^{(+,+)} = (\bar{D}^{(-,0)})^2 W^{(+,+)} = (\bar{D}^{(0,-)})^2 W^{(+,+)} = 0, \quad (5.30)$$

$$D^{(-,0)} D^{(0,-)} W^{(+,+)} = \bar{D}^{(-,0)} \bar{D}^{(0,-)} W^{(+,+)} = 0.$$

Finally, in the massless case we impose also the reality condition,

$$\widetilde{W^{(+,+)}} = W^{(+,+)}. \quad (5.31)$$

The solution of these constraints can be obtained from (4.29) by putting the central charges to zero, $z = \bar{z} = 0$ and taking into account the reality (5.31),

$$\begin{aligned} W^{(+,+)} = & u_{[i}^{(+,0)} u_{j]}^{(0,+)} i f^{ij} + i \theta^{(+,0)\alpha} \psi_{\alpha}^i u_i^{(0,+)} - i \theta^{(0,+)\alpha} \psi_{\alpha}^i u_i^{(+,0)} \\ & + i \bar{\theta}_{\dot{\alpha}}^{(+,0)} \bar{\psi}^{i\dot{\alpha}} u_i^{(0,+)} - i \bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{\psi}^{i\dot{\alpha}} u_i^{(+,0)} \\ & - 2 \theta_{\alpha}^{(+,0)} \bar{\theta}_{\dot{\alpha}}^{(+,0)} \sigma^{m\alpha\dot{\alpha}} \partial_m f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,+)} - 2 \theta_{\alpha}^{(0,+)} \bar{\theta}_{\dot{\alpha}}^{(0,+)} \sigma^{m\alpha\dot{\alpha}} \partial_m f^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,-)} \\ & - (\theta_{\alpha}^{(+,0)} \bar{\theta}_{\dot{\alpha}}^{(0,+)} + \theta_{\alpha}^{(0,+)} \bar{\theta}_{\dot{\alpha}}^{(+,0)}) \sigma^{m\alpha\dot{\alpha}} \partial_m f^{ij} (u_{[i}^{(+,0)} u_{j]}^{(-,0)} - u_{[i}^{(0,+)} u_{j]}^{(0,-)}) \\ & + \theta_{\alpha}^{(+,0)} \theta_{\beta}^{(0,+)} F^{(\alpha\beta)} + \bar{\theta}_{\dot{\alpha}}^{(+,0)} \bar{\theta}_{\dot{\beta}}^{(0,+)} \bar{F}^{(\dot{\alpha}\dot{\beta})} \\ & + i (\theta^{(+,0)} \sigma^m \bar{\theta}^{(0,+)} + \theta^{(0,+)} \sigma^m \bar{\theta}^{(+,0)}) \varepsilon_{mnrs} G^{mrs} \\ & - 2 \theta^{(+,0)\alpha} \theta^{(0,+)\beta} \bar{\theta}^{(+,0)\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^m \partial_m \psi_{\beta}^i u_i^{(-,0)} - 2 \theta^{(+,0)\beta} \theta^{(0,+)\alpha} \bar{\theta}^{(0,+)\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^m \partial_m \psi_{\beta}^i u_i^{(0,-)} \\ & - 2 \bar{\theta}^{(+,0)\dot{\alpha}} \bar{\theta}^{(0,+)\dot{\beta}} \theta^{(+,0)\alpha} \sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{\psi}_{\dot{\beta}}^i u_i^{(-,0)} - 2 \bar{\theta}^{(+,0)\dot{\beta}} \bar{\theta}^{(0,+)\dot{\alpha}} \theta^{(0,+)\alpha} \sigma_{\alpha\dot{\alpha}}^m \partial_m \bar{\psi}_{\dot{\beta}}^i u_i^{(0,-)} \\ & + 4 i \theta_{\alpha}^{(+,0)} \theta_{\beta}^{(0,+)} \bar{\theta}_{\dot{\alpha}}^{(+,0)} \bar{\theta}_{\dot{\beta}}^{(0,+)} \sigma^{m\alpha\dot{\alpha}} \sigma^{n\beta\dot{\beta}} \partial_m \partial_n f^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,-)}, \end{aligned} \quad (5.32)$$

where all components depend on the analytic coordinates (4.23). Owing to the reality condition (5.31) we have

$$\bar{f}^{ij} = f^{ij}, \quad \overline{\psi_{\alpha}^i} = \bar{\psi}_{i\dot{\alpha}}, \quad \overline{F_{(\alpha\beta)}} = \bar{F}_{(\dot{\alpha}\dot{\beta})}, \quad \overline{G^{mnr}} = G^{mnr}. \quad (5.33)$$

These components satisfy free equations of motion,

$$\square f^{ij} = 0 \quad (f^{ij} \Omega_{ij} = 0) \quad 5 \text{ real scalars}, \quad (5.34)$$

$$\sigma^{m\alpha}{}_{\dot{\alpha}} \partial_m \psi_{\alpha}^i = 0, \quad \sigma^m{}_{\alpha}{}^{\dot{\alpha}} \partial_m \bar{\psi}_{\dot{\alpha}}^i = 0 \quad 4 \text{ Weyl spinors}, \quad (5.35)$$

$$\sigma^{m\alpha}{}_{\dot{\alpha}} \partial_m F_{(\alpha\beta)} = 0, \quad \sigma^m{}_{\alpha}{}^{\dot{\alpha}} \partial_m \bar{F}_{(\dot{\alpha}\dot{\beta})} = 0 \quad 1 \text{ real Maxwell field}, \quad (5.36)$$

$$\partial_m G^{mnr} = 0, \quad \varepsilon_{mnrs} \partial^m G^{mrs} = 0 \quad 1 \text{ antisymmetric tensor field}. \quad (5.37)$$

It is well known that the antisymmetric tensor field (5.37) is classically equivalent to the scalar field on-shell. Indeed, introducing the field

$$L_m = \frac{1}{6} \varepsilon_{mnrs} G^{mrs}, \quad (5.38)$$

the equations (5.37) can be rewritten as

$$\partial^m L_m = 0, \quad \partial_m L_n - \partial_n L_m = 0. \quad (5.39)$$

A general solution of (5.39) is expressed through the scalar field subject to the Klein-Gordon equation,

$$L_m = \partial_m \phi, \quad \square \phi = 0. \quad (5.40)$$

As a result, we see that the superfield (5.32) describes the $\mathcal{N}=4$ vector multiplet, in which one of the scalars is dualized and represented by the antisymmetric tensor field. Here G^{mnr} is a strength of the antisymmetric tensor field which can always be expressed through its potential as $G^{mnr} = \partial^m B^{nr} + \partial^n B^{rm} + \partial^r B^{mn}$.

In conclusion of this subsection we comment briefly on the other ways of separation of constraints (5.4c) on the examples of the superfields $W^{(+,-)}$, $W^{(-,+)}$. Similarly as in the massive case, these superfields are constrained by (4.35,4.36) and (4.40) respectively. The component structure can be read from (4.39) by putting the central charges to zero, $z = \bar{z} = 0$. Therefore, these superfields also describe the $\mathcal{N}=4$ vector multiplet (5.34)–(5.37) in which one of the scalars is represented by the antisymmetric tensor field.

5.2 Gravitino multiplet

Let us consider the separation of constraints (5.4d) and require the superfield $\Phi^{(0,+)}$ to be annihilated by the following Grassmann derivatives,

$$\bar{D}_{\dot{\alpha}}^{(+,0)} \Phi^{(0,+)} = \bar{D}_{\dot{\alpha}}^{(0,+)} \Phi^{(0,+)} = \bar{D}_{\dot{\alpha}}^{(-,0)} \Phi^{(0,+)} = D_{\alpha}^{(0,+)} \Phi^{(0,+)} = 0. \quad (5.41)$$

The harmonic derivatives commuting with the spinor derivatives in (5.41) are $D^{(++,0)}$, $D^{(--,0)}$, $D^{(0,++)}$, $D^{(+,+)}$, $D^{(-,+)}$. Therefore we require them to annihilate the state,

$$D^{(++,0)} \Phi^{(0,+)} = D^{(--,0)} \Phi^{(0,+)} = D^{(0,++)} \Phi^{(0,+)} = D^{(+,+)} \Phi^{(0,+)} = D^{(-,+)} \Phi^{(0,+)} = 0. \quad (5.42)$$

Finally, the superfield satisfies the linearity conditions (5.8),

$$\begin{aligned} (\bar{D}^{(0,-)})^2 \Phi^{(0,+)} &= (D^{(+,0)})^2 \Phi^{(0,+)} = (D^{(-,0)})^2 \Phi^{(0,+)} = (D^{(0,-)})^2 \Phi^{(0,+)} = 0, \\ D^{(-,0)} D^{(0,-)} \Phi^{(0,+)} &= D^{(+,0)} D^{(0,-)} \Phi^{(0,+)} = D^{(+,0)} D^{(-,0)} \Phi^{(0,+)} = 0. \end{aligned} \quad (5.43)$$

To solve the constraints (5.41) we pass to the chiral-analytic coordinates,

$$y^m = x^m + i\theta^{(-,0)} \sigma^m \bar{\theta}^{(+,0)} - i\theta^{(+,0)} \sigma^m \bar{\theta}^{(-,0)} - i\theta^{(0,-)} \sigma^m \bar{\theta}^{(0,+)} - i\theta^{(0,+)} \sigma^m \bar{\theta}^{(0,-)}, \quad (5.44)$$

in which the derivatives in (5.41,5.42) are given by (as usual, we omit here the terms acting on $\Phi^{(0,+)}$ trivially)

$$\bar{D}_{\dot{\alpha}}^{(\pm,0)} = \pm \frac{\partial}{\partial \bar{\theta}^{(\mp,0)\dot{\alpha}}}, \quad \bar{D}_{\dot{\alpha}}^{(0,+)} = \frac{\partial}{\partial \bar{\theta}^{(0,-)\dot{\alpha}}}, \quad D^{(0,+)} = \frac{\partial}{\partial \theta^{(0,-)\alpha}}, \quad (5.45)$$

$$D_A^{(\pm\pm,0)} = D^{(\pm\pm,0)} + \theta_{\alpha}^{(\pm,0)} \frac{\partial}{\partial \theta_{\alpha}^{(\mp,0)}}, \quad D_A^{(0,++)} = D^{(0,++)} - 2i\theta^{(0,+)} \sigma^m \bar{\theta}^{(0,+)} \partial_m, \quad (5.46)$$

$$D_A^{(\pm,+)} = D^{(\pm,+)} - 2i\theta^{(\pm,0)} \sigma^m \bar{\theta}^{(0,+)} \partial_m \pm \theta_{\alpha}^{(0,+)} \frac{\partial}{\partial \theta_{\alpha}^{(\mp,0)}}. \quad (5.47)$$

Since the derivatives (5.45) are short, the constraints (5.41) are manifestly solved by

$$\Phi^{(0,+)} = \Phi^{(0,+)}(y^m, \theta_\alpha^{(+,0)}, \theta_\alpha^{(-,0)}, \theta_\alpha^{(0,+)}, \bar{\theta}_{\dot{\alpha}}^{(0,+)}, u). \quad (5.48)$$

Using the expressions for the harmonic derivatives (5.46,5.47) one can easily check that the following component expression for $\Phi^{(0,+)}$ solves (5.42),

$$\begin{aligned} \Phi^{(0,+)} = & f^i u_i^{(0,+)} - \theta^{(+,0)\alpha} \psi_\alpha^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,+)} + \theta^{(-,0)\alpha} \psi_\alpha^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,+)} \\ & + \theta^{(0,+)\alpha} \rho_\alpha - \frac{1}{2} \theta^{(0,+)\alpha} \psi_\alpha^{ij} (u_{[i}^{(+,0)} u_{j]}^{(-,0)} - u_{[i}^{(0,+)} u_{j]}^{(0,-)}) + \bar{\theta}_{\dot{\alpha}}^{(0,+)} \bar{\lambda}^{\dot{\alpha}} \\ & + \theta_{(\alpha}^{(+,0)} \theta_{\beta)}^{(-,0)} F^{i\alpha\beta} u_i^{(0,+)} - \theta_{(\alpha}^{(+,0)} \theta_{\beta)}^{(0,+)} F^{i\alpha\beta} u_i^{(-,0)} + \theta_{(\alpha}^{(-,0)} \theta_{\beta)}^{(0,+)} F^{i\alpha\beta} u_i^{(+,0)} \\ & + 2i\theta^{(+,0)} \sigma^m \bar{\theta}^{(0,+)} \partial_m f^i u_i^{(-,0)} - 2i\theta^{(-,0)} \sigma^m \bar{\theta}^{(0,+)} \partial_m f^i u_i^{(+,0)} \\ & + 2i\theta^{(0,+)} \sigma^m \bar{\theta}^{(0,+)} \partial_m f^i u_i^{(0,-)} \\ & + \theta^{(+,0)\alpha} \theta^{(-,0)\beta} \theta^{(0,+)\gamma} T_{(\alpha\beta\gamma)} - 2i\theta^{(+,0)\alpha} \theta^{(-,0)\beta} \bar{\theta}^{(0,+)\dot{\alpha}} \sigma_{(\alpha\dot{\alpha}}^m \partial_m \rho_\beta \\ & - i\theta^{(+,0)\alpha} \theta^{(-,0)\beta} \bar{\theta}^{(0,+)\dot{\alpha}} \sigma_{(\alpha\dot{\alpha}}^m \partial_m \psi_\beta^{ij} (u_{[i}^{(+,0)} u_{j]}^{(-,0)} - u_{[i}^{(0,+)} u_{j]}^{(0,-)}) \\ & + 2i\theta^{(-,0)\alpha} \theta^{(0,+)\beta} \bar{\theta}^{(0,+)\dot{\alpha}} \sigma_{(\alpha\dot{\alpha}}^m \partial_m \psi_\beta^{ij} u_{[i}^{(+,0)} u_{j]}^{(0,-)} \\ & - 2i\theta^{(+,0)\alpha} \theta^{(0,+)\beta} \bar{\theta}^{(0,+)\dot{\alpha}} \sigma_{(\alpha\dot{\alpha}}^m \partial_m \psi_\beta^{ij} u_{[i}^{(-,0)} u_{j]}^{(0,-)} \\ & + 2i\theta^{(+,0)\alpha} \theta^{(-,0)\beta} \theta^{(0,+)\gamma} \bar{\theta}^{(0,+)\dot{\alpha}} \sigma_{\gamma\dot{\alpha}}^m \partial_m F_{\alpha\beta}^i u_i^{(0,-)}. \end{aligned} \quad (5.49)$$

All the component fields in (5.49) depend on y^m given by (5.44) and satisfy free equations of motion,

$$\begin{aligned} \square f^i &= 0 & 4 \text{ complex scalars,} \\ \sigma^{m\alpha}{}_{\dot{\alpha}} \partial_m \rho_\alpha &= 0, \quad \sigma^m{}_\alpha{}^{\dot{\alpha}} \partial_m \bar{\lambda}_{\dot{\alpha}} &= 0 & 2 \text{ Weyl spinors,} \\ \sigma^{m\alpha}{}_{\dot{\alpha}} \partial_m \psi_\alpha^{ij} &= 0 \quad (\psi_\alpha^{ij} \Omega_{ij} = 0) & 5 \text{ Weyl spinors,} \\ \sigma^{m\alpha}{}_{\dot{\alpha}} \partial_m F_{(\alpha\beta)}^i &= 0 & 4 \text{ real Maxwell fields,} \\ \sigma^{m\alpha}{}_{\dot{\alpha}} \partial_m T_{(\alpha\beta\gamma)} &= 0 & 1 \text{ gravitino.} \end{aligned} \quad (5.50)$$

As a result we obtain the $\mathcal{N}=4$ gravitino multiplet.

The other ways of separations of constraints (5.4e,5.4f,5.4g) also lead to the gravitino multiplets realized by the superfields $\Phi^{(0,-)}$, $\Phi^{(-,0)}$, $\Phi^{(+,0)}$, respectively. The constraints for these superfields can be easily read from the general expressions (5.5)–(5.8). The component structure is analogous to (5.49) with appropriate change of $U(1)$ charges.

5.3 $\mathcal{N}=4$ supergravity multiplet

Consider the separation of constraints (5.4h) on the example of chargeless superfield $\Phi^{(0,0)} \equiv \Phi$. We impose the derivatives in the second set in (5.4h) as the constraints on Φ ,

$$\bar{D}_{\dot{\alpha}}^{(+,0)} \Phi = \bar{D}_{\dot{\alpha}}^{(0,+)} \Phi = \bar{D}_{\dot{\alpha}}^{(-,0)} \Phi = \bar{D}_{\dot{\alpha}}^{(0,-)} \Phi = 0, \quad (5.51)$$

which show this superfield to be chiral,

$$\bar{D}_{i\dot{\alpha}}\Phi = 0. \quad (5.52)$$

This constraint is explicitly solved by passing to the chiral coordinates $y^m = x^m + i\theta_i\sigma^m\bar{\theta}^i$,

$$\Phi = \Phi(y^m, \theta_{i\alpha}, u). \quad (5.53)$$

Next, we impose the harmonic constraints (4.11)

$$D^{(++,0)}\Phi = D^{(0,++)}\Phi = D^{(+,+)}\Phi = D^{(-,+)}\Phi = D^{(+,-)}\Phi = 0, \quad (5.54)$$

which mean that Φ is harmonic independent,

$$\Phi(y^m, \theta_{i\alpha}, u) = \Phi(y^m, \theta_{i\alpha}). \quad (5.55)$$

Finally, there are the linearity constraints (5.8) originating from κ -symmetry (see Appendix 2 for details),

$$D^{i\alpha}D_{\alpha}^j\Phi = 0. \quad (5.56)$$

It is well known that the solution of the equation (5.56) describes the $\mathcal{N}=4$ supergravity multiplet [11, 32]. Its component structure is given by

$$\Phi = \phi + \theta_i^\alpha\psi_\alpha^i + \theta_i^\alpha\theta_j^\beta F_{(\alpha\beta)}^{[ij]} + \theta_i^\alpha\theta_j^\beta\theta_k^\gamma\varepsilon^{ijkl}T_{l(\alpha\beta\gamma)} + \theta_i^\alpha\theta_j^\beta\theta_k^\gamma\theta_l^\delta\varepsilon^{ijkl}C_{(\alpha\beta\gamma\delta)}, \quad (5.57)$$

where the component fields satisfy

$$\begin{aligned} \square\phi &= 0 & 1 \text{ complex scalar,} \\ \sigma^{m\alpha}{}_{\dot{\alpha}}\partial_m\psi_\alpha^i &= 0 & 4 \text{ Weyl spinors,} \\ \sigma^{m\alpha}{}_{\dot{\alpha}}\partial_m F_{(\alpha\beta)}^{[ij]} &= 0 & 6 \text{ Maxwell fields,} \\ \sigma^{m\alpha}{}_{\dot{\alpha}}\partial_m T_{i(\alpha\beta\gamma)} &= 0 & 4 \text{ gravitini,} \\ \sigma^{m\alpha}{}_{\dot{\alpha}}\partial_m C_{(\alpha\beta\gamma\delta)} &= 0 & 1 \text{ Weyl tensor.} \end{aligned} \quad (5.58)$$

As a result, this superfield describes the multiplet of $\mathcal{N}=4$ supergravity.

6 Applications of $USp(4)/(U(1) \times U(1))$ harmonic superspace to the $\mathcal{N}=4$ SYM model

The quantization of $\mathcal{N}=4$ harmonic superparticle has demonstrated an important role of the specific $\mathcal{N}=4$ harmonic superspace with $USp(4)/(U(1) \times U(1))$ harmonic variables⁹ and the corresponding superfields. We have seen that this superspace is naturally associated with $\mathcal{N}=4$ superparticle and therefore it is instructive to study its properties and to try to develop field theory in it.

The purpose of this section is to apply the $USp(4)/(U(1) \times U(1))$ harmonic superspace to the $\mathcal{N}=4$ SYM model and show that the superfields, obtained in quantizing the superparticle, appear naturally in the solution of $\mathcal{N}=4$ SYM constraints. We consider also some possibilities of constructing the invariant actions depending on these superfields in harmonic superspace and discuss their relevance to the $\mathcal{N}=4$ SYM model.

⁹Such harmonic variables were introduced in [25].

6.1 Harmonic superspace analysis of $\mathcal{N}=4$ SYM constraints

Let us consider standard $\mathcal{N}=4$ superspace $Z^M = \{x^m, \theta_{i\alpha}, \bar{\theta}_{\dot{\alpha}}^i\}$ with supercovariant spinor derivatives

$$D_\alpha^i = \frac{\partial}{\partial \theta_i^\alpha} + i\bar{\theta}^{\dot{\alpha}i}\sigma_{\alpha\dot{\alpha}}^m \frac{\partial}{\partial x^m}, \quad \bar{D}_{\dot{\alpha}i} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}i}} - i\theta_i^\alpha \sigma_{\alpha\dot{\alpha}}^m \frac{\partial}{\partial x^m}. \quad (6.1)$$

According to the generic procedure of superspace formulation of the extended supersymmetric models [32], one introduces the gauge connections for these derivatives,

$$D_\alpha^i \rightarrow \nabla_\alpha^i = D_\alpha^i + V_\alpha^i, \quad \bar{D}_{\dot{\alpha}i} \rightarrow \bar{\nabla}_{\dot{\alpha}i} = \bar{D}_{\dot{\alpha}i} + \bar{V}_{\dot{\alpha}i} \quad (6.2)$$

and defines the superfield strengths by the following anticommutators,

$$\{\nabla_\alpha^i, \nabla_\beta^j\} = 2\varepsilon_{\alpha\beta} W^{ij}, \quad \{\bar{\nabla}_{\dot{\alpha}i}, \bar{\nabla}_{\dot{\beta}j}\} = 2\varepsilon_{\dot{\alpha}\dot{\beta}} \bar{W}_{ij}. \quad (6.3)$$

It is well known that the following $\mathcal{N}=4$ SYM constraints put the superfield strengths on-shell [32],

$$\bar{D}_{i\dot{\alpha}} W^{jk} = \frac{1}{3}(\delta_i^j \bar{D}_{l\dot{\alpha}} W^{lk} - \delta_i^k \bar{D}_{l\dot{\alpha}} W^{lj}), \quad (6.4)$$

$$D_\alpha^i W^{jk} + D_\alpha^j W^{ik} = 0, \quad (6.5)$$

$$\overline{W^{ij}} = \bar{W}_{ij} = \frac{1}{2}\varepsilon_{ijkl} W^{kl}. \quad (6.6)$$

Let us project the strengths W^{ij} with harmonics,

$$W^{ij} \rightarrow W^{IJ} = u^I_i u^J_j W^{ij}, \quad (6.7)$$

where the indices I, J take the values (3.24). We denote these superfields also as

$$\begin{aligned} W_1 &= u_i^{(0,+)} u_j^{(0,-)} W^{ij}, & W_2 &= u_i^{(+,0)} u_j^{(-,0)} W^{ij}, \\ W^{(+,+)} &= u_i^{(+,0)} u_j^{(0,+)} W^{ij}, & W^{(-,-)} &= u_i^{(-,0)} u_j^{(0,-)} W^{ij}, \\ W^{(+,-)} &= u_i^{(+,0)} u_j^{(0,-)} W_{ij}, & W^{(-,+)} &= u_i^{(-,0)} u_j^{(0,+)} W_{ij}. \end{aligned} \quad (6.8)$$

Contracting equations (6.4,6.5) with harmonics we find a number of Grassmann analyticity constraints for these superfields,

$$\begin{aligned} D_\alpha^{(0,+)} W_1 &= D_\alpha^{(0,-)} W_1 = \bar{D}_{\dot{\alpha}}^{(+,0)} W_1 = \bar{D}_{\dot{\alpha}}^{(-,0)} W_1 = 0, \\ D_\alpha^{(+,0)} W_2 &= D_\alpha^{(-,0)} W_2 = \bar{D}_{\dot{\alpha}}^{(0,+)} W_2 = \bar{D}_{\dot{\alpha}}^{(0,-)} W_2 = 0, \\ D_\alpha^{(+,0)} W^{(+,+)} &= D_\alpha^{(0,+)} W^{(+,+)} = \bar{D}_{\dot{\alpha}}^{(+,0)} W^{(+,+)} = \bar{D}_{\dot{\alpha}}^{(0,+)} W^{(+,+)} = 0, \\ D_\alpha^{(-,0)} W^{(-,-)} &= D_\alpha^{(0,-)} W^{(-,-)} = \bar{D}_{\dot{\alpha}}^{(-,0)} W^{(-,-)} = \bar{D}_{\dot{\alpha}}^{(0,-)} W^{(-,-)} = 0, \\ D_\alpha^{(+,0)} W^{(+,-)} &= D_\alpha^{(0,-)} W^{(+,-)} = \bar{D}_{\dot{\alpha}}^{(+,0)} W^{(+,-)} = \bar{D}_{\dot{\alpha}}^{(0,-)} W^{(+,-)} = 0, \\ D_\alpha^{(-,0)} W^{(-,+)} &= D_\alpha^{(0,+)} W^{(-,+)} = \bar{D}_{\dot{\alpha}}^{(-,0)} W^{(-,+)} = \bar{D}_{\dot{\alpha}}^{(0,+)} W^{(-,+)} = 0. \end{aligned} \quad (6.9)$$

Moreover, by construction, the superfields (6.8) are annihilated by the following harmonic derivatives,

$$\begin{aligned}
D^{(++,0)}W_1 &= D^{(--,0)}W_1 = D^{(0,++)}W_1 = D^{(0,--)}W_1 = (D^{(+,+)})^2 W_1 = 0, \\
D^{(++,0)}W_2 &= D^{(--,0)}W_2 = D^{(0,++)}W_2 = D^{(0,--)}W_2 = (D^{(-,-)})^2 W_2 = 0, \\
D^{(++,0)}W^{(+,+)} &= D^{(0,++)}W^{(+,+)} = D^{(+,+)}W^{(+,+)} = D^{(+,-)}W^{(+,+)} = D^{(-,+)}W^{(+,+)} = 0, \\
D^{(--,0)}W^{(-,-)} &= D^{(0,--)}W^{(-,-)} = D^{(-,-)}W^{(-,-)} = D^{(+,-)}W^{(-,-)} = D^{(-,+)}W^{(-,-)} = 0, \\
D^{(++,0)}W^{(+,-)} &= D^{(0,--)}W^{(+,-)} = D^{(+,-)}W^{(+,-)} = D^{(+,+)}W^{(+,-)} = D^{(-,-)}W^{(+,-)} = 0, \\
D^{(--,0)}W^{(-,+)} &= D^{(0,++)}W^{(-,+)} = D^{(-,+)}W^{(-,+)} = D^{(+,+)}W^{(-,+)} = D^{(-,-)}W^{(-,+)} = 0.
\end{aligned} \tag{6.10}$$

We see that the superfield strengths W_1 , W_2 , $W^{(+,+)}$, $W^{(-,-)}$, $W^{(+,-)}$, $W^{(-,+)}$ introduced in the subsections 5.1.1, 5.1.2 satisfy the same constraints (6.9,6.10) and therefore give the solutions of $\mathcal{N}=4$ SYM constraints (6.4,6.5).

Let us now consider the reality constraint (6.6). Applying the following identities with harmonics

$$\begin{aligned}
u^{(+,0)i}u^{(-,0)j}\varepsilon_{ijkl} &= 2u_{[k}^{(0,+)}u_{l]}^{(0,-)}, \\
u^{(0,+)i}u^{(0,-)j}\varepsilon_{ijkl} &= 2u_{[k}^{(+,0)}u_{l]}^{(-,0)}, \\
u^{(+,0)i}u^{(0,-)j}\varepsilon_{ijkl} &= -2u_{[k}^{(+,0)}u_{l]}^{(0,-)}, \\
u^{(0,+)i}u^{(-,0)j}\varepsilon_{ijkl} &= -2u_{[k}^{(0,+)}u_{l]}^{(-,0)}, \\
u^{(+,0)i}u^{(0,+)j}\varepsilon_{ijkl} &= -2u_{[k}^{(+,0)}u_{l]}^{(0,+)}, \\
u^{(-,0)i}u^{(0,-)j}\varepsilon_{ijkl} &= -2u_{[k}^{(-,0)}u_{l]}^{(0,-)},
\end{aligned} \tag{6.11}$$

we find that (6.6) leads to the reality properties of superfield strengths,

$$\begin{aligned}
\widetilde{W_{1,2}} &= W_{1,2}, \quad \widetilde{W^{(+,+)}} = W^{(+,+)}, \quad \widetilde{W^{(-,-)}} = W^{(-,-)}, \\
\widetilde{W^{(+,-)}} &= W^{(-,+)}, \quad \widetilde{W^{(-,+)}} = W^{(+,-)}.
\end{aligned} \tag{6.12}$$

These conjugation rules for superfield strengths coincide with (5.20,5.31) which were previously introduced in the superparticle considerations. This establishes the correspondence between the $\mathcal{N}=4$ superfield strengths (6.3) and the superfields obtained by the superparticle quantization.

6.2 Actions in $USp(4)/(U(1) \times U(1))$ harmonic superspace

We have seen that the superfields (6.8) satisfying the constraints and equations of motion (6.9,6.10) appear naturally in the solution of $\mathcal{N}=4$ SYM constraints (6.4)–(6.6). Now we address a question, whether it is possible to construct with the help of these superfields any $\mathcal{N}=4$ invariant superfield functionals in $USp(4)/(U(1) \times U(1))$ harmonic superspace,

which can be treated as the actions of some $\mathcal{N}=4$ supersymmetric field models. For instance, similar superfields on $\mathcal{N}=4$ harmonic superspace with $SU(4)$ harmonic variables were used in the construction of different integral invariants in [12].

Each of the superfields (6.8) lies in its own analytic subspace parameterized by eight Grassmann variables, as is seen in each line in (6.9). We restrict ourself to two superfields, $W^{(+,+)}$ and W_1 , the others can be studied in a similar way.

Let us consider first $W^{(+,+)}$, which lies in the analytic subspace with the coordinates $\{x_A^m, \theta_\alpha^{(+,0)}, \theta_\alpha^{(0,+)}, \bar{\theta}_{\dot{\alpha}}^{(+,0)}, \bar{\theta}_{\dot{\alpha}}^{(0,+)}, u\}$, where x_A^m is given by (4.23). In general, one can consider the following functional depending on this superfield

$$\int d\zeta^{(-4,-4)} \mathcal{F}(W^{(+,+)}), \quad (6.13)$$

where $d\zeta^{(-4,-4)}$ is the dimensionless analytic measure defined in (A.31). Since such a function \mathcal{F} should have definite $U(1)$ charges, $s_1 = s_2 = 4$, it can be only quartic,

$$S_4 = g \int d\zeta^{(-4,-4)} (W^{(+,+)})^4, \quad (6.14)$$

where g is a coupling constant of mass dimension -4 . Using the component decomposition (5.32) for $W^{(+,+)}$ it is easy to see that the action (6.14) contains the following term in components,

$$S_4 \sim g \int d^4x F^{\alpha\beta} F_{\alpha\beta} \bar{F}^{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}} + \dots, \quad (6.15)$$

where $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ are the spinorial components of the Maxwell strength and the dots indicate the terms for all other component fields. Since the term in the rhs of (6.15) appears in the fourth order of the decomposition of the Born-Infeld action (e.g. see [35, 36]), one can treat the action (6.14) as a part of an $\mathcal{N}=4$ supersymmetric generalization of Born-Infeld theory.

Consider now the chargeless superfield W_1 in the analytic superspace with the coordinates $\{x_A^m, \theta_\alpha^{(+,0)}, \theta_\alpha^{(-,0)}, \bar{\theta}_{\dot{\alpha}}^{(0,+)}, \bar{\theta}_{\dot{\alpha}}^{(0,-)}, u\}$, where x_A^m is given by (5.15). In general, one can write the following functional with it,

$$\int d\zeta \mathcal{F}(W_1), \quad (6.16)$$

where $d\zeta$ is the analytic measure given by (A.36). Since the measure in (6.16) is chargeless, there are no restrictions on the function \mathcal{F} . Of particular interest is the quartic potential,

$$\int d\zeta (W_1)^4 \sim \int d^4x F^{\alpha\beta} F_{\alpha\beta} \bar{F}^{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}} + \dots, \quad (6.17)$$

as it may have some relations to the $\mathcal{N}=4$ supersymmetric generalization of Born-Infeld theory as mentioned above. One can easily check the presence of F^4 term in the component

structure of (6.17) by applying the component decomposition for W_1 given by (5.21). Another interesting example is given by the logarithmic potential,

$$\int d\zeta \ln(W_1/\Lambda) \sim \int d^4x \frac{F^4}{\phi^4}, \quad (6.18)$$

where Λ is some scale which makes the combination W_1/Λ dimensionless. Clearly, the expression in lhs of (6.18) is scale independent despite the manifest presence of dimensional parameter. In components, such functional reproduces the first leading term in the low-energy effective action of $\mathcal{N}=4$ SYM model (see, e.g., [33]). Note that the $\mathcal{N}=4$ superfield description given by (6.18) for such terms is even simple than the one with $\mathcal{N}=2$ superfields [33]. In the work [37] we noticed that it is much more difficult to write down similar terms within $\mathcal{N}=3$ harmonic superspace approach.

In this subsection we saw that the superfield strengths (6.8) allowed us to construct some actions with the gauge fields in the fourth power while the $\mathcal{N}=4$ superfield description for F^2 terms remains unclear. Moreover, the strengths (6.8) are on-shell objects constrained by (6.10). Therefore the classical action for $\mathcal{N}=4$ SYM model, if it admits an $\mathcal{N}=4$ superfield description, should be constructed in terms of some potentials for these superfield strengths. One such attempt is undertaken in the Appendix 3, where we introduce the analytic prepotentials and study some superfield action with it. However, this question requires further independent studies and lies beyond the frame of the present work.

7 Summary

In this paper we have studied the construction of massive and massless $\mathcal{N}=4$ superparticle models in $\mathcal{N}=4$ harmonic superspace and their quantization. The crucial point of our considerations is the use of $USp(4)$ harmonic variables since exactly this group corresponds to the R-symmetry of $\mathcal{N}=4$ superalgebra with central charge. Since the mass of the superparticle should be equal to its central charge, as required for the construction of BPS supermultiplets, both massive and massless superparticles can be studied and quantized in such an $\mathcal{N}=4$ $USp(4)/(U(1) \times U(1))$ harmonic superspace. The quantization leads straightforwardly to the superfield realization of physically interesting multiplets of $\mathcal{N}=4$ supersymmetry. Namely, in the massive case, the $\mathcal{N}=4$ massive vector multiplet is described by four analytic superfields $W^{(+,+)}$, $W^{(-,-)}$, $W^{(+,-)}$, $W^{(-,+)}$ with different types of analyticity obeying also harmonic shortness constraints, which serve as the equations of motion for these superfields. In the massless case these superfields reduce to the usual $\mathcal{N}=4$ SYM multiplet if additional reality constraints are imposed. Moreover, there are also two chargeless superfields W_1 and W_2 with specific Grassmann and harmonic shortness constraints describing the same multiplet. All these six strength superfields are shown to appear naturally in the solution of $\mathcal{N}=4$ SYM constraints with the help of $USp(4)/(U(1) \times U(1))$ harmonic variables. Apart from the $\mathcal{N}=4$ SYM multiplet, the superparticle leads to the superfield realizations of $\mathcal{N}=4$ gravitino multiplet (with highest

helicity $3/2$) and $\mathcal{N}=4$ supergravity multiplet (with highest helicity 2). These multiplets are represented by chiral-analytic and $\mathcal{N}=4$ chiral superfields in harmonic superspace, respectively, with appropriate Grassmann and harmonic constraints.

The quantization of the $\mathcal{N}=4$ harmonic superparticle shows some new possibilities for studying the $\mathcal{N}=4$ SYM theory directly in $\mathcal{N}=4$ harmonic superspace with $USp(4)/(U(1) \times U(1))$ harmonics. The $USp(4)$ group is very suitable for this purpose, particularly because of its invariant antisymmetric 2-tensor, which raises and lowers the R-symmetry indices, similarly as the ε -tensor in the $SU(2)$ group. The corresponding $USp(4)$ harmonic superspace possesses a specific conjugation generalizing the usual complex conjugation, and harmonic projections of $\mathcal{N}=4$ superfield strengths appear real under this conjugation. Therefore the $USp(4)$ harmonics seem to be very useful for studying the $\mathcal{N}=4$ SYM model in harmonic superspace. Moreover, as is sketched in the last subsection, it is very straightforward to build some gauge invariant actions in $\mathcal{N}=4$ harmonic superspace with $USp(4)$ harmonics. We propose actions which contain F^4 term in the bosonic sector. These actions are written in an analytic superspace with an integration over half of the Grassmann variables of $\mathcal{N}=4$ superspace, such that the analytic measure is dimensionless. The F^4 term may be interpreted as the quartic term in an $\mathcal{N}=4$ Born-Infeld action. It is interesting to note that it allows for a very simple scale-invariant generalization which corresponds to the leading term in the low-energy effective action of the $\mathcal{N}=4$ SYM model. However, these issues require deeper investigations.

To conclude, we have constructed the $\mathcal{N}=4$ massive and massless superparticle models and developed their quantization. As a result we found superfields in $\mathcal{N}=4$ harmonic superspace with $USp(4)/(U(1) \times U(1))$ harmonics which describe the basic on-shell $\mathcal{N}=4$ multiplets. These superfields allow one to write some $\mathcal{N}=4$ superfield actions corresponding to $\mathcal{N}=4$ SYM theory.

It would be very interesting to develop systematically the ideas presented in Appendix 3, where we show that the analytic potentials, which serve as gauge connections for the harmonic derivatives, can be used for constructing some superfield actions.

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Appendices

1 Commutation relations in $usp(4)$ algebra

Let us introduce the operators

$$T^{ij} = \Omega^{k(i} u^{j)}{}_l \frac{\partial}{\partial u^k{}_l}, \quad (\text{A.1})$$

where u^i_j are the harmonic variables (2.8). It well known, that the operators (A.1) obey the commutation relations of $usp(4)$ algebra (see, e.g., [27]),

$$[T^{ij}, T^{kl}] = \Omega^{i(k} T^{l)j} + \Omega^{j(k} T^{l)i}. \quad (\text{A.2})$$

In the present work we prefer to use the operators (2.16) which differ from (A.1) only by constants and therefore also span the $usp(4)$ algebra. Their commutation relations are given by

$$\begin{aligned}
[D^{(++,0)}, D^{(--,0)}] &= S_1, & [D^{(0,++)}, D^{(0,--)}] &= S_2, \\
[D^{(+,+)}, D^{(-,-)}] &= S_1 + S_2, & [D^{(+,-)}, D^{(-,+)}] &= S_2 - S_1, \\
[D^{(+,-)}, D^{(+,+)})] &= 2D^{(++,0)}, & [D^{(+,-)}, D^{(-,-)}] &= -2D^{(0,--)}, \\
[D^{(+,-)}, D^{(++,0)}] &= 0, & [D^{(+,-)}, D^{(--,0)}] &= -D^{(-,-)}, \\
[D^{(+,-)}, D^{(0,++)}] &= D^{(+,+)}, & [D^{(-,-)}, D^{(--,0)}] &= 0, \\
[D^{(+,+)}, D^{(++,0)}] &= 0, & [D^{(+,+)}, D^{(--,0)}] &= -D^{(-,+)}, \\
[D^{(+,+)}, D^{(0,++)}] &= 0, & [D^{(+,+)}, D^{(0,--)}] &= D^{(+,-)}, \\
[D^{(++,0)}, D^{(0,++)}] &= 0, & [D^{(++,0)}, D^{(0,--)}] &= 0, \\
[D^{(+,+)}, D^{(-,+)}] &= 2D^{(0,++)}, & [D^{(++,0)}, D^{(-,+)}] &= D^{(+,+)}, \\
[D^{(++,0)}, D^{(-,-)}] &= D^{(+,-)}, & [D^{(0,++)}, D^{(-,+)}] &= 0, \\
[D^{(0,++)}, D^{(-,-)}] &= -D^{(-,+)}, & [D^{(0,++)}, D^{(--,0)}] &= 0, \\
[D^{(-,+)}, D^{(0,--)}] &= D^{(-,-)}, & [D^{(-,+)}, D^{(-,-)}] &= 2D^{(--,0)}, \\
[D^{(+,-)}, D^{(0,--)}] &= 0, & [D^{(-,+)}, D^{(--,0)}] &= 0, \\
[D^{(-,-)}, D^{(0,--)}] &= 0, & [D^{(--,0)}, D^{(0,--)}] &= 0, \\
[S_1, D^{(++,0)}] &= 2D^{(++,0)}, & [S_2, D^{(++,0)}] &= 0, \\
[S_1, D^{(0,++)}] &= 0, & [S_2, D^{(0,++)}] &= 2D^{(0,++)}, \\
[S_1, D^{(--,0)}] &= -2D^{(--,0)}, & [S_2, D^{(--,0)}] &= 0, \\
[S_1, D^{(0,--)}] &= 0, & [S_2, D^{(0,--)}] &= -2D^{(0,--)}, \\
[S_1, D^{(+,+)}] &= D^{(+,+)}, & [S_2, D^{(+,+)}] &= D^{(+,+)}, \\
[S_1, D^{(-,-)}] &= -D^{(-,-)}, & [S_2, D^{(-,-)}] &= -D^{(-,-)}, \\
[S_1, D^{(+,-)}] &= D^{(+,-)}, & [S_2, D^{(+,-)}] &= -D^{(+,-)}, \\
[S_1, D^{(-,+)}] &= -D^{(-,+)}, & [S_2, D^{(-,+)}] &= D^{(-,+)}, \\
[S_1, S_2] &= 0. & &
\end{aligned} \quad (\text{A.3})$$

2 Comment on the $D^\alpha D_\alpha$ constraint and κ -symmetries

Here we explain the origin of the quadratic spinor constraints (5.8, 5.56, 5.30, 5.43) appearing in the massless case. All the considerations are valid for arbitrary \mathcal{N} , however only $\mathcal{N}=4$ superparticle was studied in this work.

Consider the Lagrangian of massless superparticle (5.2) with the constraints (3.12,3.14), which in the massless case read

$$p^2 \approx 0, \quad (A.4)$$

$$D_\alpha^i = -\pi_\alpha^i + ip_m(\sigma^m \bar{\theta}^i)_\alpha \approx 0, \quad \bar{D}_{i\dot{\alpha}} = \bar{\pi}_{i\dot{\alpha}} - ip_m(\theta_i \sigma^m)_{\dot{\alpha}} \approx 0. \quad (A.5)$$

The transformations of κ -symmetry (3.7) in the massless case

$$\begin{aligned} \delta_\kappa \theta_{i\alpha} &= -ip_m \sigma_{\alpha\dot{\alpha}}^m \bar{\kappa}_i^{\dot{\alpha}}, & \delta_\kappa \bar{\theta}_{\dot{\alpha}}^i &= ip_m \kappa^{i\alpha} \sigma_{\alpha\dot{\alpha}}^m, \\ \delta_\kappa x^m &= i\delta_\kappa \theta_i \sigma^m \bar{\theta}^i - i\theta_i \sigma^m \delta_\kappa \bar{\theta}^i, & \delta_\kappa e &= -4(\bar{\kappa}_{i\dot{\alpha}} \dot{\theta}^{i\dot{\alpha}} + \dot{\theta}_i^\alpha \kappa_\alpha^i). \end{aligned} \quad (A.6)$$

are generated by

$$\delta_\kappa = \kappa^{i\alpha} [\psi_{i\alpha}, \cdot]_P - \bar{\kappa}_i^{\dot{\alpha}} [\bar{\psi}_{\dot{\alpha}}^i, \cdot]_P, \quad (A.7)$$

where $[\cdot, \cdot]_P$ is the graded Poisson bracket and

$$\psi_{i\alpha} = ip_m \sigma_{\alpha\dot{\alpha}}^m \bar{D}_i^{\dot{\alpha}} \approx 0, \quad \bar{\psi}_{\dot{\alpha}}^i = -ip_m \sigma_{\alpha\dot{\alpha}}^m D^{i\alpha} \approx 0 \quad (A.8)$$

are the generators of κ -symmetry which are first-class constraints.

We point out that the transformations (A.6) leave the superparticle Lagrangian invariant ($\delta_\kappa L_1 = 0$) for arbitrary local parameters κ_α , $\bar{\kappa}_{\dot{\alpha}}$. In particular, we can take

$$\kappa_\alpha^i = k^{ij} \theta_{j\alpha}, \quad \bar{\kappa}_{i\dot{\alpha}} = \bar{k}_{ij} \bar{\theta}_{\dot{\alpha}}^j, \quad (A.9)$$

where $k^{ij} = k^{ji}$, $\bar{k}_{ij} = \bar{k}_{ji}$ are new local bosonic parameters. The transformations (A.6) read now

$$\begin{aligned} \delta_k \theta_{i\alpha} &= -i\bar{k}_{ij} p_m \sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{j\dot{\alpha}}, & \delta_k \bar{\theta}_{\dot{\alpha}}^i &= ik^{ij} p_m \theta_j^\alpha \sigma_{\alpha\dot{\alpha}}^m, \\ \delta_k x^m &= -p^m [k^{ij} \theta_i^\alpha \theta_{j\alpha} + \bar{k}_{ij} \bar{\theta}_{\dot{\alpha}}^i \bar{\theta}^{j\dot{\alpha}}], & \delta_k e &= -4(k^{ij} \dot{\theta}_i^\alpha \theta_{j\alpha} + \bar{k}_{ij} \bar{\theta}_{\dot{\alpha}}^i \dot{\theta}^{j\dot{\alpha}}). \end{aligned} \quad (A.10)$$

They are generated by the variation

$$\delta_k = k^{ij} [K_{ij}, \cdot]_P + \bar{k}_{ij} [\bar{K}^{ij}, \cdot]_P, \quad (A.11)$$

where K_{ij} , \bar{K}^{ij} are new first-class constraints,

$$K_{ij} = -ip_m \theta_{(i}^\alpha \sigma_{\alpha\dot{\alpha}}^m \bar{\pi}_{j)}^{\dot{\alpha}} \approx 0, \quad \bar{K}^{ij} = -ip_m \pi^{(i\alpha} \sigma_{\alpha\dot{\alpha}}^m \bar{\theta}^{j)\dot{\alpha}} \approx 0. \quad (A.12)$$

Of course, these constraints are not independent, but follow from (A.8) upon contractions with Grassmann variables. With the use of (A.4,A.5) the constraints (A.12) can be rewritten as

$$\bar{\pi}_{i\dot{\alpha}} \bar{\pi}_j^{\dot{\alpha}} \approx 0, \quad \pi^{i\alpha} \pi_\alpha^j \approx 0, \quad (A.13)$$

or as

$$\bar{D}_{ij} \equiv \bar{D}_{i\dot{\alpha}} \bar{D}_j^{\dot{\alpha}} \approx 0, \quad D^{ij} \equiv D^{i\alpha} D_\alpha^j \approx 0. \quad (A.14)$$

Therefore the constraints (A.14) can also be imposed on the states upon quantization.

Note also that the generators of κ -symmetries appear from the commutators of the constraints (A.5) and (A.12),

$$[K_{ij}, D_\alpha^k]_P = \psi_{(i\alpha} \delta_{j)}^k, \quad [\bar{K}^{ij}, \bar{D}_{k\dot{\alpha}}]_P = \bar{\psi}_{\dot{\alpha}}^{(i} \delta_k^{j)}. \quad (\text{A.15})$$

$$[\bar{D}_{ij}, D_\alpha^k]_P = 4\psi_{(i\alpha} \delta_{j)}^k, \quad [D^{ij}, \bar{D}_{k\dot{\alpha}}]_P = 4\bar{\psi}_{\dot{\alpha}}^{(i} \delta_k^{j)}. \quad (\text{A.16})$$

Therefore one can consider only the constraints (A.5) and (A.14) since the κ -symmetry ones (A.6) follow from their algebra.

As a result, there is a bosonic version of κ -symmetry if we choose the parameters of κ -transformations being proportional to Grassmann coordinates. However, not all the parameters k_{ij} , \bar{k}^{ij} are independent. Indeed, a symmetric $\mathcal{N} \times \mathcal{N}$ matrix has $\mathcal{N}(\mathcal{N} + 1)/2$ independent elements that is too much in comparison with the number of first-class superparticle constraints. However, there is $U(\mathcal{N})$ R-symmetry which rotates indices i, j of all objects. It is well-known, that any symmetric complex matrix k^{ij} can be brought to the diagonal form by $U(\mathcal{N})$ rotations [29],

$$uku^T = \text{diag}(d_1, \dots, d_{\mathcal{N}}), \quad u \in U(\mathcal{N}). \quad (\text{A.17})$$

After such a rotation, there are only \mathcal{N} independent real parameters in the matrix k .

We point out the importance of accounting the constraints (A.14) in the Gupta-Bleuler quantization of a superparticle despite they are not independent but follow from spinorial constraints (A.5) and κ -symmetries (A.8). Recall that the spinorial constraints (A.5) consist of $2\mathcal{N}$ first-class and $2\mathcal{N}$ second-class constraints (which can not be separated explicitly) while the κ -symmetry constraints (A.8) have effectively $2\mathcal{N}$ first-class constraints (since they are infinitely reducible). Indeed, on the example of a chiral superfield (5.55) it is easy to see that the constraints (5.7) are not sufficient and the constraint (5.56) must be imposed to achieve the correct component structure (5.57). This is a feature of Gupta-Bleuler quantization approach in which we take into account only the constraint $\bar{D}_{i\dot{\alpha}} \approx 0$ corresponding to the chiral superfield but not $D_\alpha^i \approx 0$. As a result, some of the first class constraints originating from $D_\alpha^i \approx 0$ remain unaccounted until one has not imposed the constraints (5.56).

Finally, we note that the analogous quadratic spinorial constraints (4.17) also originate from the κ -symmetry if one passes to the bosonic parameters as in (A.9).

3 F^2 term in $\mathcal{N}=4$ $USp(4)/(U(1) \times U(1))$ harmonic superspace

It is well known that the classical actions in $\mathcal{N}=2$ and $\mathcal{N}=3$ SYM models can be written in terms of unconstrained superfields (prepotentials) in harmonic superspace [3, 5]. A similar superfield formulation for $\mathcal{N}=4$ SYM model is very desirable since it may be fruitful for studying quantum aspects of this model. One can hope that the $\mathcal{N}=4$ harmonic superspace with $USp(4)/(U(1) \times U(1))$ harmonic variables may be useful for this purpose. In particular, here we propose some functional built of analytic prepotentials which gives F^2 term in components.

In principle, one can try to construct some actions in different subspaces of full $\mathcal{N}=4$ superspace. However, the superspaces with eight Grassmann variables are more promising since the corresponding integration measure is dimensionless. Indeed, if one tries to construct an action in full $\mathcal{N}=4$ superspace one needs the dimensionful constant since the corresponding measure is of dimension $+4$. Therefore we consider the analytic subspaces of $\mathcal{N}=4$ superspace which are singled out by the covariant spinor derivatives in different lines in (6.9). Our aim now is to develop the differential geometry in one of such analytic subspaces and to build possible gauge invariant superfield functionals.

Consider the analytic subspace in $\mathcal{N}=4$ harmonic superspace with coordinates $\{x_A^m, \theta_\alpha^{(+,0)}, \theta_\alpha^{(0,+)}, \bar{\theta}_{\dot{\alpha}}^{(+,0)}, \bar{\theta}_{\dot{\alpha}}^{(0,+)}, u\}$, where x_A^m is given by (4.23). As is shown in (5.27), the Grassmann derivatives

$$D_\alpha^{(+,0)}, \quad D_\alpha^{(0,+)}, \quad \bar{D}_{\dot{\alpha}}^{(+,0)}, \quad \bar{D}_{\dot{\alpha}}^{(0,+)} \quad (\text{A.18})$$

single out the analytic superfields. The set of derivatives (A.18) is invariant under the commutators with the following harmonic derivatives

$$D^{(++,0)}, \quad D^{(0,++)}, \quad D^{(+,+)}, \quad D^{(-,+)}, \quad D^{(+,-)}. \quad (\text{A.19})$$

Therefore we can introduce five analytic gauge connections for the harmonic derivatives (A.19),

$$\begin{aligned} D^{(++,0)} &\rightarrow \nabla^{(++,0)} = D^{(++,0)} + V^{(++,0)}, \\ D^{(0,++)} &\rightarrow \nabla^{(0,++)} = D^{(0,++)} + V^{(0,++)}, \\ D^{(+,+)} &\rightarrow \nabla^{(+,+)} = D^{(+,+)} + V^{(+,+)}, \\ D^{(+,-)} &\rightarrow \nabla^{(+,-)} = D^{(+,-)} + V^{(+,-)}, \\ D^{(-,+)} &\rightarrow \nabla^{(-,+)} = D^{(-,+)} + V^{(-,+)}. \end{aligned} \quad (\text{A.20})$$

Clearly, these prepotentials contain too much component fields even in the physical sector and we have to introduce some constraints. One of the possible ways to impose the off-shell constraints is to vanish the prepotentials in the last two lines in (A.20),

$$V^{(+,-)} = 0, \quad V^{(-,+)} = 0. \quad (\text{A.21})$$

We point out that it is the constraints (A.21) which do not lead to the equations of motion for the component fields, but just reduce the number of independent components. Since the derivatives $D^{(+,-)}, D^{(-,+)}$ form $su(2)$ subalgebra in (A.3), the constraints (A.21) can be naturally resolved if one uses the harmonic variables on $USp(4)/(SU(2) \times U(1))$ coset rather than $USp(4)/(U(1) \times U(1))$. However, for our considerations it is sufficient to work in the $USp(4)/(U(1) \times U(1))$ harmonic superspace with the constrained prepotentials (A.20).

The covariant derivatives (A.20) should satisfy the same algebra (A.3) even with imposed constraints (A.21). It leads to the following constraints for the prepotentials,

$$\begin{aligned} D^{(+,-)}V^{(++,0)} &= 0, \quad D^{(-,+)}V^{(0,++)} = 0, \\ D^{(+,-)}D^{(+,-)}V^{(+,+)} &= 0, \quad D^{(-,+)}D^{(-,+)}V^{(+,+)} = 0, \\ (D^{(-,+)}D^{(+,-)} + 2)V^{(+,+)} &= 0. \end{aligned} \quad (\text{A.22})$$

Moreover, the prepotentials $V^{(++ ,0)}$, $V^{(0,++)}$, $V^{(+,+)}$ can be expressed through each other and only $V^{(+ ,+)}$ is independent,

$$\begin{aligned} D^{(-,+)}V^{(++ ,0)} &= -V^{(+ ,+)}, & D^{(+,-)}V^{(+ ,+)} &= 2V^{(++ ,0)}, \\ D^{(+,-)}V^{(0,++)} &= V^{(+ ,+)}, & D^{(-,+)}V^{(+ ,+)} &= -2V^{(0,++)}. \end{aligned} \quad (\text{A.23})$$

There are also the following reality properties for the prepotentials,

$$\widetilde{V^{(+ ,+)}} = V^{(+ ,+)}, \quad \widetilde{V^{(++ ,0)}} = V^{(0,++)}, \quad \widetilde{V^{(0,++)}} = V^{(++ ,0)}. \quad (\text{A.24})$$

The analytic prepotentials define the harmonic field strengths,

$$\begin{aligned} F^{(2,2)} &= [\nabla^{(++ ,0)}, \nabla^{(0,++)}] = D^{(++ ,0)}V^{(0,++)} - D^{(0,++)}V^{(++ ,0)} + [V^{(++ ,0)}, V^{(0,++)}], \\ F^{(3,1)} &= [\nabla^{(++ ,0)}, \nabla^{(+ ,+)}] = D^{(++ ,0)}V^{(+ ,+)} - D^{(+ ,+)}V^{(++ ,0)} + [V^{(0,++)}, V^{(+ ,+)}], \\ F^{(1,3)} &= [\nabla^{(+ ,+)}, \nabla^{(0,++)}] = D^{(+ ,+)}V^{(0,++)} - D^{(0,++)}V^{(+ ,+)} + [V^{(+ ,+)}, V^{(0,++)}], \end{aligned} \quad (\text{A.25})$$

which are analytic and gauge covariant (or invariant in the Abelian case),

$$\delta F^{(2,2)} = [\lambda, F^{(2,2)}], \quad \delta F^{(3,1)} = [\lambda, F^{(3,1)}], \quad \delta F^{(1,3)} = [\lambda, F^{(1,3)}] \quad (\text{A.26})$$

under the following gauge transformations of the prepotentials

$$\delta V^{(++ ,0)} = -\nabla^{(++ ,0)}\lambda, \quad \delta V^{(0,++)} = -\nabla^{(0,++)}\lambda, \quad \delta V^{(+ ,+)} = -\nabla^{(+ ,+)}\lambda. \quad (\text{A.27})$$

Here λ is a real analytic gauge parameter constrained by

$$D^{(+,-)}\lambda = D^{(-,+)}\lambda = 0. \quad (\text{A.28})$$

Owing to the constraints (A.22) these strengths are related to each other,

$$D^{(+,-)}F^{(2,2)} = F^{(3,1)}, \quad D^{(-,+)}F^{(2,2)} = F^{(1,3)}, \quad D^{(+,-)}F^{(1,3)} = D^{(-,+)}F^{(3,1)} = -2F^{(2,2)}. \quad (\text{A.29})$$

We stress that the constraints (A.28) are purely algebraical and mean that λ depends on $USp(4)/(SU(2) \times U(1))$ harmonics. One can avoid all these constraints by using the superfields in $USp(4)/(SU(2) \times U(1))$ harmonic superspace.

Now we apply the strength superfields (A.25) to build a gauge invariant action in analytic subspace,

$$S_2 = -\text{tr} \int d\zeta^{(-4,-4)} F^{(2,2)} F^{(2,2)} = -\frac{1}{2}\text{tr} \int d\zeta^{(-4,-4)} F^{(3,1)} F^{(1,3)}, \quad (\text{A.30})$$

where $d\zeta^{(-4,-4)}$ is the analytic measure,

$$d\zeta^{(-4,-4)} = \frac{1}{2^8} d^4 x_A du (D^{(-,0)})^2 (D^{(0,-)})^2 (\bar{D}^{(-,0)})^2 (\bar{D}^{(0,-)})^2. \quad (\text{A.31})$$

The integration over harmonic variables is defined by the following rules

$$\begin{aligned} \int du \, 1 &= 1, \\ \int du \, f^{(s_1, s_2)} &= 0, \quad \text{if } s_1 \neq 0, s_2 \neq 0, \\ \int du \, (\text{irreducible harmonic tensor}) &= 0, \end{aligned} \tag{A.32}$$

where $f^{(s_1, s_2)}$ is some function of $USp(4)/(U(1) \times U(1))$ harmonic variables. The rigorous grounds for such rules of harmonic integrals are given in the book [6] for the case of $SU(2)/U(1)$ harmonic variables and in [5, 38] for $SU(3)/(U(1) \times U(1))$ harmonics. Here we just generalize these constructions for the $USp(4)/(U(1) \times U(1))$ coset.

The action (A.30) is supersymmetric and gauge invariant by construction and contain the component fields with the spins (helicity) not higher than one. In particular, it is easy to find the vector field in its component structure by considering the following term in the prepotential $V^{(+, +)}$,

$$V^{(+, +)} = \frac{i}{2} [\theta^{(+, 0)} \sigma^m \bar{\theta}^{(0, +)} + \theta^{(0, +)} \sigma^m \bar{\theta}^{(+, 0)}] A_m + \dots \tag{A.33}$$

The real vector field A_m turns into the Maxwell strength in the $F^{(2, 2)}$ superfield (in the Abelian case),

$$F^{(2, 2)} = (\theta^{(+, 0)} \sigma^m \bar{\theta}^{(+, 0)}) (\theta^{(0, +)} \sigma^n \bar{\theta}^{(0, +)}) F_{mn} + \dots \tag{A.34}$$

and leads to the Maxwell term in the action (A.30),

$$S_2 = -\frac{1}{4} \int d^4x \, F^{mn} F_{mn} + \dots \tag{A.35}$$

The dots stand here for the other component terms in the action. As a result we see that the action contains $\mathcal{N}=4$ vector multiplet in component decomposition and may have some relation to the $\mathcal{N}=4$ supergauge theory. However, one can check that the action (A.30) contains much more propagating degrees of freedom than a single $\mathcal{N}=4$ SYM multiplet. Therefore to make the action (A.30) physical one needs more superfield constraints for the prepotentials and this issue requires further studies.

Another way in the seek of unconstrained $\mathcal{N}=4$ SYM action in the $USp(4)$ harmonic superspace is the use of other analytic subspaces in $\mathcal{N}=4$ superspace. Of particular interest may be the analytic subspace with coordinates $\{x_A^m, \theta_\alpha^{(+, 0)}, \theta_\alpha^{(-, 0)}, \bar{\theta}_\alpha^{(0, +)}, \bar{\theta}_\alpha^{(0, -)}, u\}$, where x_A^m is given by (5.15). The corresponding analytic measure is chargeless,

$$d\zeta = \frac{1}{2^8} d^4x_A du (D^{(+, 0)})^2 (D^{(-, 0)})^2 (\bar{D}^{(0, +)})^2 (\bar{D}^{(0, -)})^2. \tag{A.36}$$

One can use other prepotentials and field strengths in this subspace for constructing the invariant actions. We leave these questions for further studies.

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